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# Polylogarithms and Subordination of Some Cubic Polynomials

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Abstract. Let $V_3(z,f)$ and $\sigma_3^{(1)}(z,f)$ be the cubic polynomials representing, respectively, the 3rd de la Vallée Poussin mean and the 3rd Cesàro mean of order 1 of a power series $f(z)$. If $\mathscr{K}$ denotes the usual class of convex univalent functions in the open unit disk centered at the origin, we show that, in general, $V_3(z,f) \not\prec \sigma_3^{(1)}(z,f)$, for all $f \in \mathscr{K}$. Making use of polylogarithms, we identify a transformation, $\Lambda : \mathscr{K} \to \mathscr{K}$, such that $V_3(z,\Lambda(f)) \prec \sigma_3^{(1)}(z,\Lambda(f))$ for all $f \in \mathscr{K}$. Here '$\prec$' stands for subordination between two analytic functions.

## 1. Introduction

For a real number r, $0 < r < 1$ let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ denote the open disc in the complex plane with center at the origin and radius $r$ and $\mathbb{D} := \mathbb{D}_1$. We denote by $\mathcal{A}$ the class of all analytic functions defined in $\mathbb{D}$ that are normalized by the conditions $f(0) = f'(0) - 1 = 0$, and by the class $\mathscr{S}$ of univalent functions in $\mathcal{A}$. Further, we let $\mathscr{K}$ denote the subclass of those functions in $\mathscr{S}$ that map $\mathbb{D}$ onto convex domains and

$$\mathcal{S}_2 = \{f \in \mathscr{K} : zf' \in \mathscr{K}\}.$$

Analytically, $f \in \mathscr{K}$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, z \in \mathbb{D}.$$

If $\mathscr{C}$ denotes the class of close-to-convex functions in $\mathbb{D}$, then $\mathcal{S}_2 \subset \mathscr{K} \subset \mathscr{C} \subset \mathscr{S}$.

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In 1983, Lewis [7], in an extremely involved proof, showed that the polylogarithmic functions

$$Li_\beta(z) = \sum_{n=1}^{\infty} \frac{1}{n^\beta} z^n, \quad \beta \geq 0 \tag{1.1}$$

are convex univalent in $\mathbb{D}$. For $\beta > 0$, the series in (1.1) has the integral representation given by the formula (see [13]):

$$Li_\beta(z) = \frac{1}{\Gamma(\beta)} \int_0^1 z(\log(1/t))^{\beta-1} \frac{1}{1-tz} dt, \quad |z| < 1, \ \beta > 0.$$

Here $\Gamma$ stands for Euler gamma function. Note that

$$Li_1(z) = \int_0^1 \frac{z}{1-tz} dt = -\log(1-z) \ \ , |z| < 1 \tag{1.2}$$

and

$$Li_2(z) = \int_0^1 z \log(1/t) \frac{1}{1-tz} dt = -\int_0^z \frac{\log(1-\zeta)}{\zeta} d\zeta, |z| < 1. \tag{1.3}$$

The dilogarithm, $Li_2$, is of particular importance, since it arises in many integrals that cannot be expressed in terms of elementary functions. Due to this, it is used in many computer algebra systems, such as Maple (see [1]), where it takes the form $dilog(z) = Li_2(1-z)$.

The de la Vallée Poussin means are defined by

$$v_n(f,t) = \frac{1}{2\pi} \int_0^{2\pi} w_n(t-u) f(u) du, \quad n \in \mathbb{N},$$

where $f$ is periodic real-valued function and

$$w_n(t) = \frac{2^n (n!)^2}{(2n)!} (1+\cos t)^n = \frac{1}{\binom{2n}{n}} \sum_{k=-n}^{n} \binom{2n}{n+k} e^{ikt}$$

are the de la Vallée Poussin kernels. In 1958, Pólya and Schoenberg [8] cast these means in terms of complex-valued analytic functions using the Hadamard product.

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ and is denoted by $(f * g)(z)$. If $f$ and $g$ are analytic in $\mathbb{D}$, then $f * g$ is analytic in $\mathbb{D}$ as well. Define

$$V_n(z) = \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n+k} z^k, \quad n \in \mathbb{N}, \ z \in \mathbb{D}.$$

For a given function $f(z) = \sum_{n=1}^{\infty} a_n z^n$,

$$V_n(z,f) = (V_n * f)(z) = \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$$

is the nth de la Vallée Poussin mean of $f$. Pólya and Schoenberg ([8, Theorem 2, p. 298]) proved that for all $n \in \mathbb{N}$ the de la Vallée Poussin means $V_n(z,f)$ are convex if and only if $f$ is convex.
Similarly, for a real number $\alpha \geq 0$, if

$$\sigma_n^{(\alpha)}(z) = \binom{n+\alpha-1}{n-1}^{-1} \sum_{k=1}^{n} \binom{n+\alpha-k}{n-k} z^k, \quad n \in \mathbb{N},\ z \in \mathbb{D},$$

then the polynomial,

$$\sigma_n^{(\alpha)}(z,f) = (\sigma_n^{(\alpha)} * f)(z) = \binom{n+\alpha-1}{n-1}^{-1} \sum_{k=1}^{n} \binom{n+\alpha-k}{n-k} a_k z^k,$$

is called the $n$th Cesàro mean of $f$ of order $\alpha$. Note that $\sigma_n^{(\alpha)}(z) = \sigma_n^{(\alpha)}(z, z/(1-z))$. Several authors (see [3, 4, 6, 10]) studied univalence properties of the polynomials $\sigma_n^{(\alpha)}(z, z/(1-z))$ and it is now known (see [6]) that for $\alpha \geq 1$ and $n \in \mathbb{N}$, $\sigma_n^{(\alpha)}(z, z/(1-z)) \in \mathscr{C}$. Using the fact that the class $\mathscr{C}$ is closed under convolution with convex functions (see [11, Theorem 2.2]), we immediately get that for $\alpha \geq 1$ and $n \in \mathbb{N}$, $\sigma_n^{(\alpha)}(z,f) \in \mathscr{C}$ for all $f \in \mathscr{K}$. For some more geometric and subordination properties of Cesàro means we refer to [12, 14].

Let two functions $f$ and $g$ be analytic in $\mathbb{D}_r$. Then $f$ is called subordinate to $g$, written $f \prec g$, in $\mathbb{D}_r$ if there exists an analytic function $w$ satisfying the inequality $|w(z)| \leq |z| < r$ such that $f(z) = g(w(z))$ in $\mathbb{D}_r$. If $g$ is univalent in $\mathbb{D}_r$, then $f \prec g$ in $\mathbb{D}_r$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}_r) \subset g(\mathbb{D}_r)$.

In 1981, Singh and Singh [15] proved the following result:

**Theorem 1.1.** *If $f \in \mathscr{K}$, then $V_2(z,f) \prec \sigma_2^{(1)}(z,f)$ in $\mathbb{D}$.*

Recently, the authors [18] proved that, infact, $V_2(z,f) \prec \sigma_2^{(\alpha)}(z,f)$ holds in $\mathbb{D}$ for all $f \in \mathscr{K}$ and for all $\alpha \geq 1$.

A natural question which arises is: Is $V_3(z,f) \prec \sigma_3^{(\alpha)}(z,f)$ in $\mathbb{D}$, for all $f \in \mathscr{K}$?

The primary objective of this paper is to answer this question in the case that $\alpha = 1$. The paper is arranged as follows. In Section 2, we collect some known results which we shall use in the sequel. Section 3 starts with an example showing that answer to above question is 'no'. Then, using polylogarithms, a subclass of $\mathscr{K}$ is identified such that for all $f$ in this subclass, $V_3(z,f) \prec \sigma_3^{(1)}(z,f)$ in $\mathbb{D}$.

## 2. Preliminaries

In this section we recall the following definition and results which shall be needed to prove our results in this paper.

**Definition 2.1.** A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z) = \sum_{n=1}^\infty a_n z^n$, $a_1 = 1$, is univalent and convex in $\mathbb{D}$, we have

$$\sum_{n=1}^\infty a_n b_n z^n \prec f(z).$$

**Lemma 2.2.** [16] *A sequence $\{b_n\}_1^\infty$ of complex numbers is a subordinating factor sequence if and only if*

$$\Re\left[1 + 2\sum_{n=1}^\infty b_n z^n\right] > 0, \quad z \in \mathbb{D}.$$

**Lemma 2.3.** [11] *Let $\phi$ and $\psi$ be convex functions in $\mathbb{D}$ and suppose that $f$ is subordinate to $\phi$. Then $f * \psi$ is subordinate to $\phi * \psi$ in $\mathbb{D}$.*

**Lemma 2.4.** [2] *Let $\mathcal{P}_n$ be the set of all polynomials of degree $n, n \geq 2$. Assume that $Q \in \mathcal{P}_n$ has all its critical points $\zeta_j$ in $\overline{\mathbb{D}}, j = 1, 2, \ldots, n-1$. Let $P \in \mathcal{P}_n$ satisfy $P(0) = Q(0)$ and*

$$Q(\zeta_j) \notin P(\mathbb{D}), \quad j = 1, 2, \ldots, n-1.$$

*Then $P \prec Q$ in $\mathbb{D}$.*

**Lemma 2.5.** [9] *Given a polynomial,*

$$r(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \tag{2.1}$$

*of degree $n$, let*

$$M_k = \det \begin{bmatrix} \bar{B}_k^T & A_k \\ & \\ \bar{A}_k^T & B_k \end{bmatrix} \quad (k = 1, 2, \ldots, n), \tag{2.2}$$

*where $A_k$ and $B_k$ are triangular matrices*

$$A_k = \begin{bmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ 0 & a_0 & \cdots & a_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix}, \quad B_k = \begin{bmatrix} \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_{n-k+1} \\ 0 & \bar{a}_n & \cdots & \bar{a}_{n-k+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{a}_n \end{bmatrix}.$$

*Then the polynomial $r(z)$ has all its zeros inside the unit circle $|z| = 1$ if and only if the determinants $M_1, M_2, ..., M_n$ are all positive.*

## 3. Main Results

We begin with the following example which shows that the subscript 2 in Theorem 1.1 can not be replaced with 3, in general.

**Example 3.1.** Let $I(z) = z/(1-z)$ be the right half plane mapping which maps $\mathbb{D}$ onto $\{w \in \mathbb{C} : \Re(w) > -1/2\}$. This function is a distinguished member of the class $\mathscr{K}$. We claim that

$$V_3(z, I) \not\prec \sigma_3^{(1)}(z, I)$$

in $\mathbb{D}$. We note that

$$\sigma_3^{(1)}(z, I) = z + \frac{2}{3}z^2 + \frac{1}{3}z^3$$

and

$$V_3(z, I) = \frac{3}{4}z + \frac{3}{10}z^2 + \frac{1}{20}z^3.$$

Now $\zeta_1 = (-2+i\sqrt{5})/3$, $\zeta_2 = (-2-i\sqrt{5})/3$ are the only critical points of $\sigma_3^{(1)}(z, I)$ and both lie on $|z| = 1$. As $V_3(0, f) = \sigma_3^{(1)}(0, f)$, therefore, in view of Lemma 2.4, if $V_3(z, I) \prec \sigma_3^{(1)}(z, I)$, we must have $\sigma_3^{(1)}(\zeta_j, I) \notin V_3(\mathbb{D}, I), j = 1, 2$. But this will be the case if all the zeros of the polynomials $\phi_j(z) = V_3(z, I) - \sigma_3^{(1)}(\zeta_j, I)$, $j = 1, 2$, lie outside the circle $|z| = 1$. First consider

$$\phi_1(z) = \frac{1}{20}z^3 + \frac{3}{10}z^2 + \frac{3}{4}z + \frac{38 - i10\sqrt{5}}{81}.$$

Then all the zeros of $\phi_1(z)$ will lie outside the circle $|z| = 1$ if and only if all the zeros of the polynomial

$$\phi_1\left(\frac{1}{z}\right) = \frac{1}{20} + \frac{3}{10}z + \frac{3}{4}z^2 + \left(\frac{38 - i10\sqrt{5}}{81}\right)z^3$$

lie inside the circle $|z| = 1$. Now comparing $\phi_1(1/z)$ with the polynomial $r(z)$ (with $n = 3$) in (2.1) we note that

$$a_0 = \frac{1}{20}, \quad a_1 = \frac{3}{10}, \quad a_2 = \frac{3}{4}, \quad a_3 = \frac{38 - i10\sqrt{5}}{81}.$$

It is easy to verify that $M_1$ and $M_2$ as given by (2.2) are positive. We now calculate

$$M_3 = \det \begin{bmatrix} \bar{B_3}^T & A_3 \\ \bar{A_3}^T & B_3 \end{bmatrix},$$

where

$$A_3 = \begin{bmatrix} \frac{1}{20} & \frac{3}{10} & \frac{3}{4} \\ 0 & \frac{1}{20} & \frac{3}{10} \\ 0 & 0 & \frac{1}{20} \end{bmatrix}, \quad B_3 = \begin{bmatrix} \frac{38+i10\sqrt{5}}{81} & \frac{3}{4} & \frac{3}{10} \\ 0 & \frac{38+i10\sqrt{5}}{81} & \frac{3}{4} \\ 0 & 0 & \frac{38+i10\sqrt{5}}{81} \end{bmatrix}.$$

Using *Mathematica* [17], we get

$$\begin{aligned} M_3 &= \det \begin{bmatrix} \frac{38-i10\sqrt{5}}{81} & 0 & 0 & \frac{1}{20} & \frac{3}{10} & \frac{3}{4} \\ \frac{3}{4} & \frac{38-i10\sqrt{5}}{81} & 0 & 0 & \frac{1}{20} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{4} & \frac{38-i10\sqrt{5}}{81} & 0 & 0 & \frac{1}{20} \\ \frac{1}{20} & 0 & 0 & \frac{38+i10\sqrt{5}}{81} & \frac{3}{4} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{20} & 0 & 0 & \frac{38+i10\sqrt{5}}{81} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{10} & \frac{1}{20} & 0 & 0 & \frac{38+i10\sqrt{5}}{81} \end{bmatrix} \\ &\approx -0.00145985. \end{aligned}$$

Since $M_3 < 0$, by Lemma 2.5 we get, $\phi_1(1/z)$ does not have all its zeros inside the circle $|z| = 1$. This implies that $\sigma_3^{(1)}(\zeta_1, I) \in V_3(\mathbb{D}, I)$. Similarly, we can prove that $\sigma_3^{(1)}(\zeta_2, I) \in V_3(\mathbb{D}, I)$. Hence, by Lemma 2.4, $V_3(z, I) \not\prec \sigma_3^{(1)}(z, I)$, $z \in \mathbb{D}$.

For geometric illustration of this example, we have drawn boundaries of the domains $V_3(\mathbb{D}, I)$ and $\sigma_3^{(1)}(\mathbb{D}, I)$ in Figure 1. The zoomed portion of this figure clearly shows that $V_3(z, I) \not\prec \sigma_3^{(1)}(z, I)$.

**Definition 3.2.** We define a class of functions, $\mathcal{K}_\beta$, as under:

$$\mathcal{K}_\beta := \{g : g(z) = (Li_\beta * f)(z),\ f \in \mathscr{K}, \beta \geq 0\}.$$

For $\beta \geq 0$, as $Li_\beta$ is convex and convolution of two convex functions is convex, so, $\mathcal{K}_\beta \subset \mathscr{K}$. Obviously, in view of (1.2) and (1.3), we have

$$\mathcal{K}_1 = \left\{ \int_0^z \frac{f(\zeta)}{\zeta} d\zeta : f \in \mathscr{K}, |z| < 1 \right\}$$

and

$$\mathcal{K}_2 = \left\{ \int_0^z \log\left(\frac{z}{\zeta}\right) \frac{f(\zeta)}{\zeta} d\zeta : f \in \mathscr{K}, |z| < 1 \right\}. \tag{3.1}$$

**Lemma 3.3.** *Let $Li_\beta$ be given by (1.1). Then the cubic polynomial $\sigma_3^{(1)}(z, Li_\beta)$ is convex univalent in $\mathbb{D}$ for all $\beta \geq \beta_0$, where $\beta_0$ ($\approx 1.039$) is the positive root of the transcendental equation*

$$2^{2\beta+1}(3^{\beta+2} - 27) + 3^\beta(25 - 3^{\beta+2}) = 0. \tag{3.2}$$

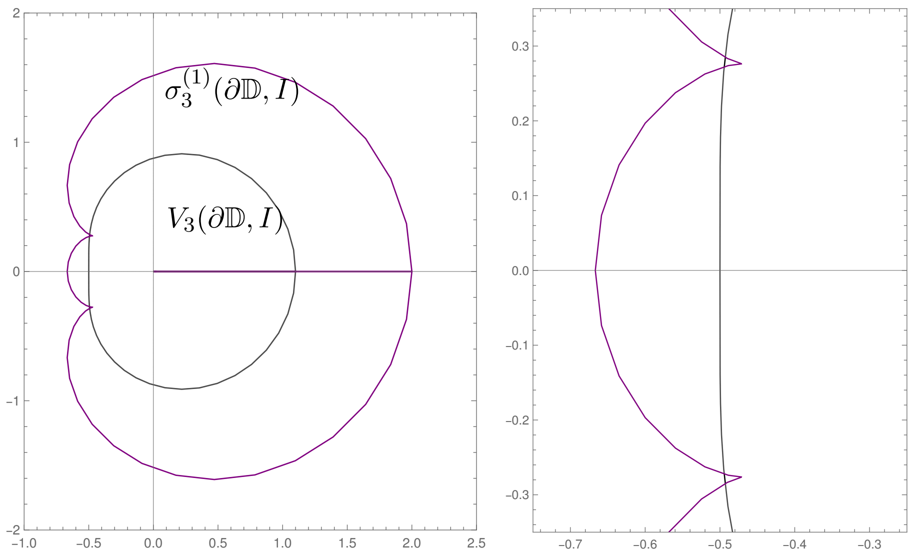


Figure 1: $V_3(\partial\mathbb{D}, I) \nsubseteq \sigma_3^{(1)}(\partial\mathbb{D}, I)$

*Proof.* We have $\sigma_3^{(1)}(z, Li_\beta) = z + \frac{2}{3}\,\frac{z^2}{2^\beta} + \frac{z^3}{3^{1+\beta}} = g(z)$ (say).
Then

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) = \Re\left(\frac{\frac{z^2}{3^{\beta-1}} + \frac{8}{3}\frac{z}{2^\beta} + 1}{\frac{z^2}{3^\beta} + \frac{4}{3}\frac{z}{2^\beta} + 1}\right) = \frac{\Re\left((\frac{z^2}{3^{\beta-1}} + \frac{8}{3}\frac{z}{2^\beta} + 1)(\frac{\overline{z}^2}{3^\beta} + \frac{4}{3}\frac{\overline{z}}{2^\beta} + 1)\right)}{|\frac{z^2}{3^\beta} + \frac{4}{3}\frac{z}{2^\beta} + 1|^2}.$$

Thus $g$ will be convex univalent in $\mathbb{D}$ provided $\Re(t(z)) \geq 0$ in $\mathbb{D}$, where

$$t(z) = \left(\frac{z^2}{3^{\beta-1}} + \frac{8}{3}\frac{z}{2^\beta} + 1\right)\left(\frac{\overline{z}^2}{3^\beta} + \frac{4}{3}\frac{\overline{z}}{2^\beta} + 1\right).$$

Putting $z = e^{i\theta}, 0 \leq \theta < 2\pi$ , we get

$$T(\theta) = \Re[t(e^{i\theta})] = 1 + \frac{32}{9}\,\frac{1}{4^\beta} + \frac{3}{9^\beta} - \frac{4}{3^\beta} + \frac{4\cos\theta}{2^\beta}\left(1 + \frac{5}{3^{\beta+1}}\right) + \frac{8}{3^\beta}\cos^2\theta.$$

We minimize $T(\theta)$, for $\theta \in [0, 2\pi)$. Note that

$$T_\theta(\theta) = -4\sin\theta\left[\frac{1}{3}\,\frac{5}{6^\beta} + \frac{1}{2^\beta} + \frac{4\cos\theta}{3^\beta}\right],$$

and, so, critical points of $T(\theta)$ are given by

$$\sin\theta = 0, \qquad 4\cos\theta = -\left(\frac{3}{2}\right)^\beta\left(1 + \frac{5}{3^{\beta+1}}\right).$$

It is now easy to verify that $T(\theta)$ has global minimum value

$$2^{2\beta+1}\,9^{\beta+1}(3^\beta-1)\left[2^{2\beta+1}(3^{\beta+2}-27)+3^\beta(25-3^{\beta+2})\right]$$

at the critical point given by

$$4\cos\theta=-\left(\frac{3}{2}\right)^\beta\left(1+\frac{5}{3^{\beta+1}}\right).$$

Using *Mathematica* [17], we conclude that $min\ T(\theta)\geq 0$ for $\beta\geq\beta_0$ ($\approx 1.039$), where $\beta_0$ is the positive root of (3.2). As, $\Re\left(1+zg''(z)/g'(z)\right)=1$ at $z=0$, using minimum principle for harmonic functions, we conclude that $g(z)=\sigma_3^{(1)}(z,Li_\beta)$ maps $\mathbb{D}$ univalently onto a convex domain for $\beta\geq\beta_0$. □

Graph of transcendental function in (3.2) is shown in Figure 2 below.

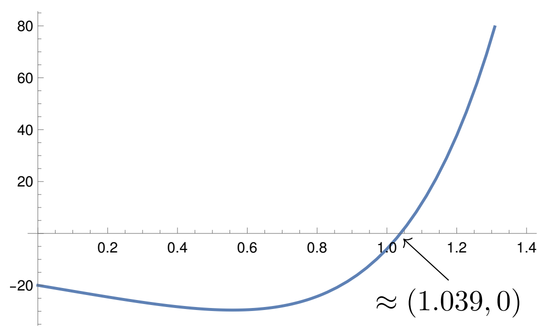


Figure 2: Graph of transcendental function in eq(3.2)

**Lemma 3.4.** *If $Li_\beta$ is given by (1.1), then*

$$V_3(z,Li_\beta)\prec\sigma_3^{(1)}(z,Li_\beta),$$

*in $\mathbb{D}$ for all $\beta>\beta_0$, where $\beta_0$ is same as in Lemma 3.3.*

*Proof.* As $Li_\beta(z)=z+\sum\limits_{n=2}^{\infty}(z^n/n^\beta)$, we have

$$\sigma_3^{(1)}(z,Li_\beta)=z+\frac{2}{3}\,\frac{z^2}{2^\beta}+\frac{1}{3}\,\frac{z^3}{3^\beta},$$

and

$$V_3(z,Li_\beta)=\frac{3}{4}z+\frac{3}{10}\,\frac{z^2}{2^\beta}+\frac{1}{20}\,\frac{z^3}{3^\beta}.$$

In view of Lemma 3.3 and Definition 2.1, it suffices to show that the sequence

$\{3/4, 9/20, 3/20, 0, 0, ...\}$ is a subordinating factor sequence. Applying Lemma 2.2, this will be true if

$$\Re(H(z)) > 0, z \in \mathbb{D},$$

where

$$H(z) = 1 + 2\left(\frac{3}{4}z + \frac{9}{20}z^2 + \frac{3}{20}z^3\right).$$

Putting $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, in $H(z)$ and then taking the real part we get

$$\begin{aligned} \Re(H(e^{i\theta})) &= 1 + \frac{3}{2}\left(\cos\theta + \frac{3}{5}\cos 2\theta + \frac{1}{5}\cos 3\theta\right) \\ &= \frac{1}{10}\left[1 + 3\left(2\cos\theta + 6\cos^2\theta + 4\cos^3\theta\right)\right]. \end{aligned} \tag{3.3}$$

On calculating the critical points, and by simple calculations, we can easily verify that (3.3) has minimum value at $\cos\theta = -1/(3+\sqrt{3})$ and this minimum value is equal to $(3-\sqrt{3})/30 = 0.042...$, which is certainly greater than zero. As $\Re(H(z)) = 1$ at $z = 0$, so we conclude that $\Re(H(z)) > 0$ in $\mathbb{D}$, by minimum principle for harmonic functions. □

In Figure 3, we have drawn boundaries of the domains $V_3(\mathbb{D}, Li_\beta)$ and $\sigma_3^{(1)}(\mathbb{D}, Li_\beta)$, taking $\beta = 2$.

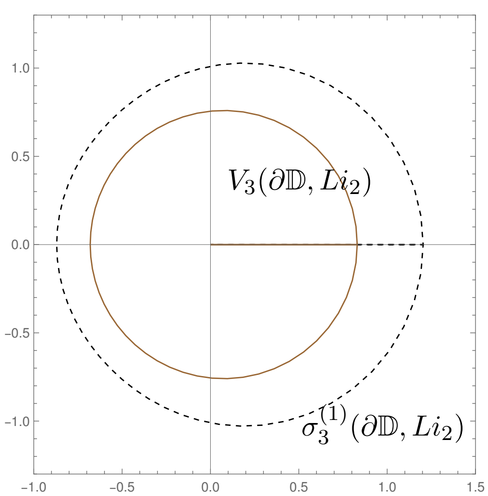


Figure 3: $V_3(\partial\mathbb{D}, Li_2) \subset \sigma_3^{(1)}(\partial\mathbb{D}, Li_2)$

**Theorem 3.5.** *Let $\beta_0$ be the number as in* Lemma 3.3. *Then for all real numbers $\beta$, $\beta \geq \beta_0$, and for all $f \in \mathscr{K}$, we have,*

$$V_3(z, Li_\beta * f) \prec \sigma_3^{(1)}(z, Li_\beta * f),$$

*in $\mathbb{D}$; or, equivalently,*

$$V_3(z, g) \prec \sigma_3^{(1)}(z, g),$$

*in $\mathbb{D}$ for all $g \in \mathcal{K}_\beta$ $(\beta \geq \beta_0)$.*

*Proof.* By Lemma 3.3, $\sigma_3^{(1)}(z, Li_\beta)$ is convex univalent in $\mathbb{D}$ for all $\beta \geq \beta_0$. The proof, therefore, follows from Lemma 2.3 and Lemma 3.4 and observing that $\sigma_3^{(1)}(z, Li_\beta * f) = \sigma_3^{(1)}(z, Li_\beta) * f$. □

If we choose $\beta = 2$ in above Theorem 3.5, we get, in view of (3.1), the following result:

**Corollary 3.6.** *For all $f \in \mathscr{K}$,*

$$V_3\left(z, \int_0^z \log\left(\frac{z}{\zeta}\right)\frac{f(\zeta)}{\zeta}d\zeta\right) \prec \sigma_3^{(1)}\left(z, \int_0^z \log\left(\frac{z}{\zeta}\right)\frac{f(\zeta)}{\zeta}d\zeta\right),$$

*in $\mathbb{D}$.*

If $f \in \mathcal{S}_2$, then $zf' \in \mathscr{K}$ and $Li_\beta * zf' = Li_{\beta-1} * f$. Thus we get:

**Corollary 3.7.** *For all real numbers $\beta$, $\beta \geq \beta_0$, and for all $f \in \mathcal{S}_2$, we have,*

$$V_3(z, Li_{\beta-1} * f) \prec \sigma_3^{(1)}(z, Li_{\beta-1} * f),$$

*in $\mathbb{D}$. Here, the number $\beta_0$ is same as in* Lemma 3.3.

Taking $\beta = 2$ in Corollary 3.7 and noting that $Li_1 * f = -\log(1-z) * f = \int_0^z (f(\zeta)/\zeta)\, d\zeta$, we obtain:

**Corollary 3.8.** *For all $f \in \mathcal{S}_2$, we have,*

$$V_3\left(z, \int_0^z \frac{f(\zeta)}{\zeta}d\zeta\right) \prec \sigma_3^{(1)}\left(z, \int_0^z \frac{f(\zeta)}{\zeta}d\zeta\right),$$

*in $\mathbb{D}$.*

**Remark 3.9.** In 1990, Komatu [5] introduced the integral operator, $\mathcal{L}_c^\beta : \mathcal{A} \to \mathcal{A}$, by

$$\mathcal{L}_c^\beta[f](z) := \frac{(1+c)^\beta}{\Gamma(\beta)} \int_0^1 \left(\log\frac{1}{t}\right)^{\beta-1} t^{c-1} f(tz)dt \quad (c > -1, \beta \geq 0).$$

It is easy to verify that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $\mathcal{L}_c^{\beta}[f]$ defined above can be expressed by the series expansion as follows:

$$\mathcal{L}_c^{\beta}[f](z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\beta} a_n z^n.$$

In view of (1.1), we obtain:

$$\mathcal{L}_0^{\beta}[f](z) = (Li_{\beta} * f)(z).$$

Therefore, Theorem 3.5 can be restated as under:

**Theorem 3.10.** *For all real numbers $\beta$, $\beta \geq \beta_0$, and for all $f \in \mathscr{K}$, we have*

$$V_3\left(z, \mathcal{L}_0^{\beta}[f]\right) \prec \sigma_3^{(1)}\left(z, \mathcal{L}_0^{\beta}[f]\right),$$

*in $\mathbb{D}$, where the number $\beta_0$ is as in* Lemma 3.3.

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