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Polynomial-Filled Function Algorithm for Unconstrained Global Optimization Problems

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ABSTRACT.
The filled function method is useful in solving unconstrained global optimization problems. However, depending on the type of function, and parameters used, there are limitations that cause difficultiies in implemenations. Exponential and logarithmic functions lead to the overflow effect, requiring iterative adjustment of the parameters. This paper proposes a polynomial-filled function that has a general form, is non-exponential, non-logarithmic, non-parameteric, and continuously differentiable. With this newly proposed filled function, the aforementioned shortcomings of the filled function method can be overcome. To confirm the superiority of the proposed filled function algorithm, we apply it to a set of unconstrained global optimization problems. The data derived by numerical implementation shows that the proposed filled function can be used as an alternative algorithm when solving unconstrained global optimization problems.

1. Introduction

The filled function method was introduced in [6] in 1993; it was introduced to correct the limitations faced by many traditional methods, such as the tunneling function method, the covering method, and the branch and bound method. A filled function works algorithmically to move from one local minimum point to a lower local minimum point through three main steps. First, the objective function is minimized. Second, a new function, called the filled function, is built at the

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local minimum point found in Step 1 , and local minimum of it is found. Third, the local minimum point of the filled function found in Step 2 is used as an initial point to minimize the objective function. The capability of the filled function algorithm broke with the notion that solving unconstrained global optimization problems should always involve the use of a multi-start approach or a partitioning method in the search domain. However, along with the development of the method, the use of exponentials and parameters in the filled function [16, 17, 14, 7] led to many unexpected issues.

A filled function involving an exponential function approaches infinity rapidly; this causes the the overflow effect. This phenomenon makes the graph of the filled function almost the same as its tangent line. Thus, the local minimum point of the filled function is essentially a pseudo-minimizer. The rationale behind the use of exponential or logarithmic functions is the stretching effect of these two functions, especially in the region $\chi_1(x^*) = \{x \in \chi : g(x) \geq g(x^*)\} \setminus \{x^*\}$, where $g(x)$ is a cost function, x^* is a local minimum point of $g(x)$ in χ, and χ is an operation domain. By this effect, the cost function $g(x)$ has no local minimum points in $\chi_1(x^*)$. As a result, many filled functions still employ either an exponential term or a logarithmic function.

On the other hand, the use of parameters is aimed at enabling the constructed filled function to meet three conditions stated in the definition of the filled function. The first condition states that if the filled function is built at a local minimum point x^* of a cost function $g(x)$, then the filled function attains its local maximum at x^*. The second condition asserts that for all $\chi_1(x^*)$, x is not a stationary nor a saddle point of the filled function constructed at x^*. The last condition requires that if the filled function is created at x^*, and x^* is not a global minimum, then the set $\chi_2(x^*) = \{x \in \chi : g(x) < g(x^*)\}$ contains at least one local minimum point of the filled function.

Parametric-filled functions were studied in [15, 5, 3]. Though the parameters add some difficulty, the advantages that can provide in computational performance in the implementation phase, means that they should not be ignored. For instance, the filled function formulated by [3] utilized an inverse trigonometric function to eliminate the overflow effect produced by using exponential functions. Filled functions constructed to have a general form were studied in [7, 4, 9, 19]. Having a general form, allows a variety of functions to be used to solve optimization problems.However, it introduces the challenge of determining parameter values during the computational stage.

From previous discussions, we tentatively concluded that parameters should be eliminated for the filled function method to be superior. Such efforts were first attempted by [2]. The idea was to select a different function on sets $\chi_1(x^*)$ and $\chi_2(x^*)$. In $\chi_1(x^*)$, the function is made independent from the cost function and a descent function. Thus, the local minimum points in $\chi_1(x^*)$ are all eliminated when minimizing the filled function. On the other hand, in the set $\chi_2(x^*)$, the property of the formed filled function is influenced by the cost function. An's filled function was defined as follows:

$$\omega_1(x, x^*) = -\text{sign}(g(x) - g(x^*)) \|x - x^*\|^2, \tag{1.1}$$

where

$$\text{sign}(\ell) = \begin{cases} 1, & \ell \geq 0 \\ -1, & \ell < 0 \end{cases}.$$

From (1.1), $\omega_1(x, x^*)$ is discontinuous at points where $g(x) = g(x^*)$. However, this property limits the kind of local minimization procedure one can use on the filled function. If a non-gradient based methodology is implemented, such a property becomes beneficial. However, such a method requires a high computational effort. Another problem arises from $\|x - x^*\|^2$. The norm causes the function value to increase uncontrollably (overflow). This undesirable effect is then reduced by changing $\|x - x^*\|^2$ into $\arctan\left(\|x - x^*\|^2\right)$ by [10]. To increase the number of local minimization methods that can be employed, the authors in [8] offered a continuously differentiable filled function defined as follows:

$$\omega_2(x, x^*) = -\|x - x^*\|^2 \beta(g(x) - g(x^*)) \tag{1.2}$$

where

$$\beta(s) = \begin{cases} 1 & s \geq 0 \\ -e^{s^2+2} & s < 0 \end{cases}.$$

However, an exponential function is still involved in (1.2). Therefore, the undesired characteristic previously discussed can possibly occur during the computational stage. The author of [1] attempted to present a new type of parameter filled function as follows:

$$\omega_3(x, x^*) = \frac{1}{1 + \|x - x^*\|^2} \beta(g(x) - g(x^*)) \tag{1.3}$$

where

$$\beta(s) = \begin{cases} \pi/2 & s \geq 0 \\ \pi/2 - \arctan(s^2) & s < 0 \end{cases}.$$

However, in our analysis, the following term

$$\frac{1}{1 + \|x - x^*\|^2}$$

in (1.3) has almost the same effect as the exponential function, i.e., the change in value is as fast as that in the exponential function.

We propose a new polynomial-filled function to accommodate the need for an effective and efficient parameter-free filled function algorithm. The proposed filled function is simple, does not involve any parameters, and is continuously differentiable. The proposed filled function will be formed in a general form. The filled function was constructed with a polynomial form because polynomial functions are simpler than other filled functions, which generally use exponential, logarithmic, inverse trigonometric, or other transcendental functions. Consequently, the proposed algorithm, where one of the phases involves filled functions, is expected to become more efficient than other filled function algorithms.

This paper is organized as follows. Section 2 provides the preliminaries, assumptions, and definitions involved in this study. Section 3 introduces a new parameter-free polynomial-filled function and its analytical properties. Section 4 discusses a global minimum algorithm, where the proposed polynomial-filled function is involved in one of the algorithm steps. Section 5 presents the numerical experiments and comparisons with some recently filled function algorithms. Finally, Section 6 offers conclusions drawn from the study.

2. Preliminaries

The unconstrained global optimization problems solved in this article should have a solution, i.e., the global minimum value of the cost function. As the problem is unconstrained, the global minimum point must be found in R^n. However, from the numerical point of view, yielding a global minimum value in R^n is impossible. To guarantee the existence of a global minimum point, we assumed that the cost function $g(x)$ is coercive, that is:

$$\lim_{\|x\|\to+\infty} g(x) = +\infty.$$

The coercive property of $g(x)$ implies the existence of a closed bounded set χ exists, such that $\chi = \chi_1 \cup \chi_2 \cup \{x^*\}$, with

$$\chi_1 = \{x \in \chi : g(x) \geq g(x^*)\} \setminus \{x^*\}$$

and

$$\chi_2 = \{x \in \chi : g(x) < g(x^*)\},$$

where x^* is a local minimum point of $g(x)$. Therefore, the unconstrained global optimization displayed in Problem 1 could be transformed into Problem 2.

Problem 1. *Minimize cost function $g(x)$, where $x \in R^n$.*

Problem 2. *Minimize cost function $g(x)$, where $x \in \chi$.*

In conclusion, Problems (1) and (2) are equivalent, and the global solution of Problem (2) can be considered the global solution in R^n.

In this paper, $g(x)$ possibly has infinite local minimum points but only finite local minimum points that have different values. The cost function $g(x)$ is a first-order and continuously differentiable function. There are various definition of filled functions in the literature. We use the following.

Definition 1. *[18]. A real valued function $\omega(x, x^*)$ is a* filled function *of a cost function $g(x)$ at x^*, where x^* is a local minimum point of $g(x)$ in χ if it satisfies the following properties.*

1. *The point x^* is a strict local maximum point of $\omega(x, x^*)$.*
2. *If $x \in \chi_1$, then x is not the stationary point of $\omega(x, x^*)$.*
3. *If $\chi_2 \neq \emptyset$, then a local minimum point x' of $\omega(x, x^*)$ exists in χ_2.*

3. New Filled Function and Its Properties

The parameter-free, continuously differentiable filled functions considered in [8], [11], and [12], give several idea for creating filled functions as defined in the previous section. One idea is that the filled function should be a piecewise function, allowing one to select a function, such as $-\|x - x^*\|^n$ where $n \geq 1$ is an integer, with a descent direction property in the region χ_1. As the polynomial-filled function formed in this study is intended to be continuously differentiable, one would consider only even values of n. The next task is to find other functions. However, the selected functions ensure that the formed filled function is continuous at x, where $g(x) = g(x^*)$. For example, $-s + 1$, $s^2 + 1$, $-s^{2n+1} + s^{2n} + 1$, with $n \geq 1$. This rationale was used to create the polynomial-filled function

$$\omega(x, x^*) = \ell_1(\|x - x^*\|^\alpha)\,\ell_2(g(x) - g(x^*)) \tag{3.1}$$

where α is an even integer, such that $\alpha \geq 2$, and $\ell_2 : R \to R$ is a single real valued function with $\ell_2(s) = 1$ for $s \geq 0$ and $\ell_2(s) = \lambda(s)$ for $s < 0$. The condition on α will make $\omega(x, x^*)$ continuously differentiable. Therefore, we obtained the following:

$$\omega(x, x^*) = \ell_1(\|x - x^*\|^\alpha),$$

for all $x \in \chi_1$, and

$$\omega(x, x^*) = \ell_1(\|x - x^*\|^\alpha)\,\lambda(g(x) - g(x^*)),$$

for all $x \in \chi_2$.
To build a specific general function (3.1), functions ℓ_1 and λ should satisfy some properties:

1. $\ell_1(s)$ and $\lambda(s)$ are polynomial functions.
2. ℓ_1 is continuously differentiable.
3. $\ell_1(0) = 0$.
4. $\ell_1(s) < 0$ for all $s \in (0, \infty)$.
5. $\ell_1'(s) \leq 0$ in $[0, \infty)$.

6. λ is continuously differentiable in $(-\infty, 0)$.
7. $\lambda(s) > 1$ for all $s \in (-\infty, 0)$.
8. $\lambda'(s) < 0$ at $s \in (-\infty, 0)$.
9. $\lim_{s \to 0^-} \lambda(s) = 1$.

We need to prove that the function in (3.1) satisfies the three properties in Definition (1).

Theorem 1. *Point x^*, which is the local minimum point of $g(x)$, is a strict local maximum point of $\omega(x, x^*)$.*

Proof. x^* is a local minimum point of $g(x)$, which implies the existence of an open ball $\mathrm{B}(x^*, \sigma)$, such that $g(x) \geq g(x^*)$ for all $x \in \mathrm{B}(x^*, \sigma) \cap \chi$. Since $g(x) \geq g(x^*)$, then for all $x \in \mathrm{B}(x^*, \sigma) \subset \chi_1$, the value of the polynomial-filled function is as follows

$$\omega(x, x^*) = \ell_1(\|x - x^*\|^\alpha).$$

Given $\|x - x^*\|^\alpha > 0$ and from property (4), the following is obtained:

$$\omega(x, x^*) = \ell_1(\|x - x^*\|^\alpha) < 0.$$

From Property (3) of ℓ_1, we determined the following:

$$\omega(x^*, x^*) = \ell_1(\|x^* - x^*\|^\alpha) = 0.$$

Hence,

$$\omega(x, x^*) < \omega(x^*, x^*),$$

for all $x \in \mathrm{B}(x^*, \sigma) \cap \chi$. Therefore, x^* is a strict local maximum point of $\omega(x, x^*)$ in χ. □

The minimization process of the filled function in the filled function algorithm requires x^* to be at the top of the basin of attraction of $\omega(x, x^*)$. Theorem 1 proves this property. The next two theorems are provided to show that $\omega(x, x^*)$ has no stationary point in χ. Theorem 2 is a necessary condition for a filled function to have no stationary points.

Theorem 2. *Assume that (1) x^* is a local minimum point of $g(x)$, (2) x^M and x^N are elements of χ_1, and (3) $\|x^M - x^*\| < \|x^N - x^*\|$. Then,*

$$\omega(x^N, x^*) < \omega(x^M, x^*).$$

Proof. As x^M and x^N are elements of χ_1, $g(x^M) \geq g(x^*)$ and $g(x^N) \geq g(x^*)$, respectively. From the definition of the proposed filled function,

$$\omega(x^M, x^*) = \ell_1\left(\|x^M - x^*\|^\alpha\right)$$

and

$$\omega\left(x^N, x^*\right) = \ell_1\left(\left\|x^N - x^*\right\|^\alpha\right).$$

The difference between the value of ω at x^M and x^N is as follows:

$$\omega\left(x^N, x^*\right) - \omega\left(x^M, x^*\right) = \ell_1\left(\left\|x^N - x^*\right\|^\alpha\right) - \ell_1\left(\left\|x^M - x^*\right\|^\alpha\right).$$

From properties (4) and (5) of ℓ_1, function ℓ_1 is decreasing, and the value is negative. Moreover, from the assumption of the theorem, $\left\|x^M - x^*\right\| < \left\|x^N - x^*\right\|$. Therefore, the following inequality holds:

$$\ell_1\left(\left\|x^N - x^*\right\|^\alpha\right) < \ell_1\left(\left\|x^M - x^*\right\|^\alpha\right).$$

Hence, the consequence is $\omega\left(x^N, x^*\right) < \omega\left(x^M, x^*\right)$. This condition proves the theorem. □

Theorem 2 reveals that the proposed filled function is a descent territory in χ_1, and it is needed because the filled function algorithm minimizes ω. However, some local minimization procedures require a zero gradient. Hence, the proposed filled function should not have any stationary points in χ_1.

Theorem 3. *If x^* is a local minimum point of $g(x)$, then χ_1 does not contain the stationary points of $\omega(x, x^*)$.*

Proof. Assuming that $x^M \in \chi_1$, the following can be obtained:

$$d^T \nabla\omega\left(x^M, x^*\right) < 0.$$

As $x^M \in \chi_1$, then $g\left(x^M\right) \geq g\left(x^*\right)$. Therefore, the value of ω at x^M is as follows:

$$\omega\left(x^M, x^*\right) = \ell_1\left(\left\|x^M - x^*\right\|^\alpha\right).$$

The gradient of ω at x^M is given by the following:

$$\nabla\omega\left(x^M, x^*\right) = \alpha{\ell'}_1\left(\left\|x^M - x^*\right\|^\alpha\right)\left\|x^M - x^*\right\|^{\alpha-2}\left(x^M - x^*\right).$$

Given that x^* is the element of interior of χ, $d = \left(x^M - x^*\right)$ is a feasible direction. The directional derivative of ω at x^M is computed as follows:

$$d^T \nabla\omega\left(x^M, x^*\right) = \alpha{\ell'}_1\left(\left\|x^M - x^*\right\|^\alpha\right)\left\|x^M - x^*\right\|^\alpha.$$

As $\alpha \geq 2$, $\left\|x^M - x^*\right\|^\alpha > 0$, and from property (5) of ℓ_1, the following is achieved:

$${\ell'}_1\left(\left\|x^M - x^*\right\|^\alpha\right) < 0.$$

Thus, $d^T \nabla\omega\left(x^M, x^*\right) < 0$. This result proves the theorem. □

Thus far, Theorems 1-3 have proven that the proposed filled function satisfies the first and second axioms required by Definition 1.

Theorem 4. *Assume that x^* is a local minimum point of $g(x)$. If $\chi_2 \neq \emptyset$, then $\omega(x, x^*)$ has a local minimum point in χ_2.*

Proof. Assume that $\bar{\chi}_2 = \{x \in \chi : g(x) \leq g(x^*)\}$. As $\bar{\chi}_2 \subset \chi$, then $\bar{\chi}_2$ is bounded. Hence, $\bar{\chi}_2$ is a compact and non-empty set. From the form of the proposed filled function, Equation (3.1) is continuously differentiable. From the Weirstrass theorem, $\omega(x, x^*)$ has a local minimum point $\tilde{x}^* \in \bar{\chi}_2$. The proposed filled function $\omega(x, x^*)$ is differentiable at $\tilde{x}^*$. Thus, $\tilde{x}^*$ is a stationary point of $\omega(x, x^*)$. On the other hand, $\nabla\omega(\tilde{x}^*, x^*) = 0$. From Theorems 2 - 3, $\omega(x, x^*)$ does not have stationary points in the region, such that $g(x) = g(x^*)$, and χ_2 is non-empty. Thus, $\tilde{x}^* \in \bar{\chi}_2$. □

Theorem 4 proves that $\omega(x, x^*)$ satisfies the last condition of Definition 1. These theorems guarantee that the global minimum algorithm can be performed. The following theorem is given as an additional property of the proposed filled function (3.1).

Theorem 5. *Assume that x^* is a local minimum point of $g(x)$. If $x^N \in \chi_1$ and $x^M \in \chi_2$, such that $\left\|x^N - x^*\right\| < \left\|x^M - x^*\right\|$, then $\omega\left(x^M, x^*\right) < \omega\left(x^N, x^*\right)$.*

Proof. As $x^N \in \chi_1$, the value of the polynomial-filled function is defined as follows:

$$\omega\left(x^N, x^*\right) = \ell_1\left(\left\|x^N - x^*\right\|^{\alpha}\right).$$

On the other hand, as $x^M \in \chi_2$, based on (3.1), the value of the polynomial-filled function at x^M is given as follows:

$$\omega\left(x^M, x^*\right) = \ell_1\left(\left\|x^M - x^*\right\|^{\alpha}\right)\lambda\left(g\left(x^M\right) - g\left(x^*\right)\right).$$

Given that $x^M \in \chi_2$, then $g\left(x^M\right) - g\left(x^*\right) < 0$. Based on the properties of λ, $\lambda\left(g\left(x^M\right) - g\left(x^*\right)\right) > 1$ for all x^M. Properties (4) and (5) of ℓ_1 reveal that $\ell_1\left(\left\|x^M - x^*\right\|^{\alpha}\right)$ is negative and strictly decreasing. With

$$\left\|x^N - x^*\right\| < \left\|x^M - x^*\right\|,$$

the following can be obtained:

$$\ell_1\left(\left\|x^M - x^*\right\|^{\alpha}\right) < \ell_1\left(\left\|x^N - x^*\right\|^{\alpha}\right).$$

From Equation (4), $\alpha \geq 2$. Therefore, we have the following:

$$\ell_1\left(\left\|x^M - x^*\right\|^{\alpha}\right)\lambda\left(g\left(x^M\right) - g\left(x^*\right)\right) < \ell_1\left(\left\|x^N - x^*\right\|^{\alpha}\right).$$

Thus, $\omega\left(x^M, x^*\right) < \omega\left(x^N, x^*\right)$. □

4. Filled Function Algorithm

This section focuses on the global minimum algorithm. The following algorithm will be employed to solve the given global optimization problems in this paper.

Poly-ffm Algorithm

Step 1. This step is intended to select a certain quantity. First, the initial point x_0 is selected from the feasible domain χ. Second, a small real number τ, which is usually $0 < \tau < 1$, is obtained. Third, set $l = 1$.

Step 2. This phase minimizes the cost function $g(x)$ by any local minimum procedure. In our study, the BFGS method was employed. This method, which is based on the literature, has a high efficiency. In this step, it yields the first local minimum point x^*.

Step 3. The local minimum point x^* yielded in Step 2 is employed to create the initial point to minimize the proposed filled function, i.e., x_l^i, with $i = 1, 2, \ldots, p$, where $p \geq 2n$, and n is the number of dimension of the cost function. The initial points are formed as $x_l^i = x^* + \tau e_i$, and e_i is the coordinate direction.

Step 4. The value of i starts from 1.

Step 5. In this step, if $i \leq p$, then the algorithm will proceed to Step 6. If all values of i have been used to minimize $g(x)$ using the initial points $x_l^i = x^* + \tau e_i$, but no better local minimum point of $g(x)$ has been obtained, then the algorithm will be terminated, and x^* will be considered as the global minimum point of $g(x)$ in χ.

Step 6. The initial point x_l^i will be examined in this step. If x_l^i is contained in χ, then the algorithm proceeds to Step 7. However, if x_l^i is outside the set χ, then we let $i = i + 1$, and the algorithm proceeds to Step 5.

Step 7. In this step, the proposed filled function is constructed at x^*:

$$\omega\left(x, x^*\right) = \ell_1\left(\|x - x^*\|^{\alpha}\right) \ell_2\left(g\left(x\right) - g\left(x^*\right)\right).$$

Step 8. The minimization process of the proposed filled function $\omega\left(x, x^*\right)$ is carried out in this step using the initial points $x_l^i = x^* + \tau e_i$. The local minimum of $\omega\left(x, x^*\right)$ obtained is denoted by x'.

Step 9. This step examines the local minimum point x'. If x' is contained in χ, and $g(x') < g(x^*)$, then we set $l = l + 1$, $x_0 = x'$, and the algorithm will return to Step 2. However, if these two conditions do not satisfy the conditions, we set $i = i + 1$ and proceed to Step 5.

5. Numerical Experiment

As the proposed filled function is in general form, in the implementation, we used one of the specific filled functions, which can be categorized as Equation (4), as follows:

$$\omega(x, x^*) = -\|x - x^*\|^2 \ell_2 (g(x) - g(x^*)),$$

where

$$\ell_2(s) = \begin{cases} 1 & s \geq 0 \\ -s + 1 & s < 0 \end{cases}.$$

The nine steps of Poly-ffm algorithm were implemented to solve the benchmark unconstrained global optimization problems as follows:

Probem 1: Three-hump back camel function

$$g(x) = -1.05x_1^4 + 2x_1^2 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2.$$

This cost function is minimized in the interior of the box:

$$-3 \leq x_j \leq 3,$$

where $j = 1, 2$. This cost function has a single global minimum point at $x^* = (0, 0)$. Its global minimum value is $g(x^*) = 0$.

Probem 2: Six-hump back camel function

$$g(x) = -2.1x_1^4 + 4x_1^2 + \frac{1}{3}x_1^6 - x_1x_2 + 4x_2^4 - 4x_2^2$$

This cost function is minimized in the interior of the box:

$$-3 \leq x_j \leq 3,$$

where $j = 1, 2$. This problem has two global minimum points, namely, $x^* = (0.0898, 0.7126)$ and $x^* = (-0.0898, -0.7126)$, with $g(x^*) = -1.0316$.

Probem 3: Rastrigin function

$$g(x) = -\cos(18x_1) + x_1^2 - \cos(18x_2) + x_2^2$$

This cost function is minimized in the interior of the box

$$-1 \leq x_j \leq 1,$$

where $j = 1, 2$. Rastrigin function achieves its global minimum at $x^* = (0, 0)$ where $g(x^*) = -2$.

Probem 4: Two-dimensional function

$$g(x) = u^2 + v^2,$$

where

$$u = 1 - x_1 + c \sin(4\pi x_2) - 2x_2$$

and

$$v = -0.5 \sin(2\pi x_1) + x_2,$$

with $c = 0.2$, $c = 0.5$, and $c = 0.05$.
This cost function is minimized in the interior of the box

$$-10 \leq x_j \leq 10,$$

where $j = 1, 2$. This problem has a global minimum value $g(x^*) = 0$.

Probem 5: Treccani function

$$g(x) = 4x_1^2 + 4x_1^3 + x_1^4 + x_2^2$$

This cost function is minimized in the interior of the box:

$$-3 \leq x_j \leq 3,$$

where $j = 1, 2$. This problem has a global minimum value $g(x^*) = 0$.

Probem 6: Shubert function

$$g(x) = u.v,$$

where

$$u = \sum_{i=1}^{5} i \cos[(i+1)x_1 + i]$$

and

$$v = \sum_{i=1}^{5} i \cos[(i+1)x_2 + i].$$

This cost function is minimized in the interior of the box

$$0 \leq x_j \leq 10,$$

where $j = 1, 2$. This global optimization problem has 760 local minimum points where the global minimum value is $g(x^*) = -186.7309$.

Probem 7: n-Dimensional function

$$g(x) = \frac{\pi}{n}(u + v + w),$$

with

$$u = 10\sin^2 \pi x_1,$$

$$v = \sum_{i=1}^{n-1} (x_i - 1)^2 \left(1 + 10\sin^2 \pi x_{i+1}\right),$$

and

$$w = (x_n - 1)^2.$$

This cost function is minimized in the interior of the box

$$-10 \leq x_j \leq 10,$$

where $j = 1, 2, \ldots, n$. This cost functions attain its global minimum value, which is $g(x^*) = 0$, at $x^* = (1, \ldots, 1)$.

Probem 8: n-Dimensional Rastrigin function

$$g(x) = 10n + \sum_{i=1}^{n} \left[x_i^2 - 10\cos(2\pi x_i)\right].$$

This cost function is minimized in the interior of the box

$$-5.12 \leq x_j \leq 5.12,$$

where $j = 1, 2, ..., n$. This function achieves its global minimum point at $x^* = (0, \ldots, 0)$, $g(x^*) = 0$.

Problems 1–8 will be solved by the Poly-ffm algorithm. The results are displayed in Tables 1–11. In the tables, t indicates the number of local minimum points of $g(x)$ in χ obtained by the Poly-ffm algorithm. The last t indicates the global local minimum point. For $t = 1$, x_1^0 is the initial point to execute the Poly-ffm algorithm, and for $(t = 2, 3, \ldots)$, x_t^0 is the local minimum point of the proposed filled function.

Tables 1–11 illustrate some of the results obtained by the Poly-ffm algorithm. The findings indicate that the proposed filled function is reliable to solve unconstrained global optimization problems. Comparison should be performed to examine the competitiveness of the Poly-ffm algorithm. The accuracy of the global minimum value of the cost function should be considered in the comparison stage. The recent filled function algorithm offered by [13] (we call it the ffm algorithm) was selected. The comparison is given as follows.

Table 1: Results of Problem 1.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(1.8883, 2.4348)$	$(1.7476, 0.8738)$	0.2986
2	$(-0.3211, -0.3920)$	$(0.0060e-11, -0.2869e-11)$	8.4103e-24

Table 2: Results of Problem 2.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(-2.3651, 1.5669)$	$(-1.6071, 0.5687)$	2.1043
2	$(0.2332, -0.7941)$	$(-0.0898, -0.7127)$	-1.0316

Table 3: Results of Problem 3.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(0.3897, -0.3658)$	$(0.3469, -0.3469)$	-1.7578
2	$(0.3469, 0.0038)$	$(0.3469, -0.0000)$	-1.8789
3	$(-0.0038, 0.0000)$	$(-0.1428e-18, -0.0157e-18)$	-2

Table 4: Results of Problem 4, with $c = 0.2$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(7.5774, -8.2346)$	$(8.7341, -3.3355)$	8.8414
2	$(0.0756, 0.5876)$	$(1.0175, 0.0548)$	1.7660e-17

Table 5: Results of Problem 4, with $c = 0.5$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(7.6552, -6.5510)$	$(7.8000, -6.5850)$	72.5124
2	$(-3.8865, 0.4091)$	$(1.0000, -0.0000)$	1.4348e-19

Table 6: Results of Problem 5.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(1.1690, -1.0974)$	$(0.0038e-08, 0.1341e-08)$	1.8033e-18

Table 7: Results of Problem 6.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(6.1165, -3.4712)$	$(6.6174, -2.5109)$	-13.8031
2	$(6.0535, -3.0302)$	$(6.0878, -3.0032)$	-30.7808
3	$(4.8338, -1.9942)$	$(4.8581, -2.0072)$	-79.4109
4	$(5.5216, -1.3918)$	$(5.4829, -1.4251)$	-186.7309

Table 8: Results of Problem 7 with $n = 2$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(5.3103, 5.9040)$	$(4.9594, 5.9968)$	64.1238
2	$(4.9594, 0.9967)$	$(4.9594, 1.0000)$	24.8793
3	$(0.9587, 1.0000)$	$(1.0000, 1.0000)$	8.2195e-16

Table 9: Results of Problem 7 with $n = 3$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(-2.4363, 4.0868, 4.5903)$	$(-1.9697, 2.9977, 4.9899)$	30.2267
2	$(1.0135, -0.7409, 4.9867)$	$(1.0000, 1.0000, 1.0000)$	6.7045e-20

Table 10: Results of Problem 8 with $n = 2$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(1.5648, 2.0799)$	$(1.9899, 0.9950)$	4.9748
2	$(-0.0120, -0.0060)$	$(-0.2408e-10, 0.5000e-10)$	0

Table 11: Results of Problem 8 with $n = 3$.

t	x_t^0	x_t^*	$g(x_t^*)$
1	$(-0.6573, -2.3235, 2.5430)$	$(-1.9899, 0.0000, 2.9849)$	12.9344
2	$(0.0125, 0.0000, 2.9849)$	$(-0.0000, 0.0000, 2.9849)$	8.9546
3	$(0.0000, 0.0000, -0.0167)$	$(0.0000, 0.0000, 0.2489e-09)$	0

Table 12: Comparison of the results.

Problem	Poly-ffm algorithm	ffm algorithm
	$g(x^*)$	$g(x^*)$
1	8.4103e-24	1.3537e-15
4 ($c = 0.2$)	1.7660e-17	6.4583e-16
4 ($c = 0.5$)	1.4348e-19	2.3665e-15
5	1.8033e-18	2.3139e-16
7 ($n = 2$)	8.2195e-16	1.4720e-14
7 ($n = 3$)	6.7045e-20	5.7060e-14

From Table 12, the Poly-ffm algorithm yields more accurate results than the algorithm given in [13].

6. Conclusion

This paper proposed a general form of polynomial-filled functions, where neither exponential nor logarithmic functions are involved. These filled functions were employed in the global optimization algorithm called Poly-ffm algorithm. Eights cost functions, which are commonly used as test functions to examine the effectiveness of an algorithm, were solved by the Poly-ffm algorithm. The numerical data yielded from the experimental computation showed that our study successfully obtained the global minimum values of the given cost functions. Further, comparison was performed to reveal the accuracy of the global minimum value obtained by the Poly-ffm algorithm with another typical filled function algorithm. The comparison results revealed the that global minimum values yielded by Poly-ffm algorithm were more accurate.

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