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The Gallai and Anti-Gallai Graphs of Strongly Regular Graphs

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Abstract. In this paper, we show that if G is strongly regular then the Gallai graph $\Gamma(G)$ and the anti-Gallai graph $\Delta(G)$ of G are edge-regular. We also identify conditions under which the Gallai and anti-Gallai graphs are themselves strongly regular, as well as conditions under which they are 2-connected. We include also a number of concrete examples and a discussion of spectral properties of the Gallai and anti-Gallai graphs.

1. Introduction

Given a graph G, the *line graph* $L(G)$ of G has the edges of G as its vertices, with two vertices adjacent in $L(G)$ if the corresponding edges have a vertex in common in G. The line graph $L(G)$ can then be decomposed into the Gallai graph $\Gamma(G)$ and anti-Gallai graph $\Delta(G)$, as follows. The *Gallai graph* $\Gamma(G)$ of G has the edges of G as its vertices, with two vertices adjacent in $\Gamma(G)$ if their corresponding edges have a vertex in common in G but do not lie on a common triangle in G. The *anti-Gallai graph* $\Delta(G)$ of G again has the edges of G as its vertices, with two vertices in $\Delta(G)$ adjacent if their corresponding edges lie on a common triangle in G. We will refer to G as the *underlying graph* of the graphs $L(G), \Gamma(G)$, and $\Delta(G)$.

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Clearly, $L(G)$ is the edge-disjoint union of $\Gamma(G)$ and $\Delta(G)$. Line graphs have been extensively studied in the literature, however in comparison the theory of Gallai and anti-Gallai graphs has attracted less attention since their introduction in [15]. Good overviews on the topic can be found in [21, 22], while more recent papers include [1, 3, 4, 16, 19, 23, 25, 26]. Furthermore, anti-Gallai graph have found application in linguistics, specifically in identifying polysemous words [2]. This provides motivation for examining the spectral properties of anti-Gallai graphs, particularly in relation to the spectra of their underlying graphs. The spectra of some classes of Gallai and anti-Gallai graphs have already been studied, in [24].

Our interest in this paper is in furthering the study of Gallai and anti-Gallai graphs, particularly in relating spectral and other properties of these graphs with those of their underlying ones. However, there is an issue which presents significant difficulties. Regularity properties of G do not always translate to similar properties of $\Gamma(G)$ and $\Delta(G)$. In particular, G being regular does not imply that $\Gamma(G)$ and $\Delta(G)$ are also regular. Contrast this with the situation for line graphs, where regularity of G implies regularity of $L(G)$, a fact which has greatly facilitated the study of line graphs. However, a key observation in this paper is that $\Gamma(G)$ and $\Delta(G)$ of a strongly regular graph G are regular, and in fact are even edge-regular. This has prompted us to focus on the Gallai and anti-Gallai graphs of strongly regular graphs, and this is the topic of this paper.

We will now define strongly regular graphs. Let G be a k-regular graph with n vertices. The graph G is said to be *strongly regular* [6] with parameters (n, k, λ, μ) if the following conditions hold:

(1) G is neither complete nor empty;
(2) any two adjacent vertices of G have λ common neighbours;
(3) any two non-adjacent vertices of G have μ common neighbours.

We note that the case $\mu = 0$ corresponds to a disjoint union of complete graphs; this is not an interesting case for our purposes, and so we will assume henceforth that $\mu > 0$, which implies that G has diameter 2. We note further that the cycles C_4 and C_5 are strongly regular, however all questions that we will consider reduce to trivialities in these cases, so we will henceforth assume $k \geq 3$ whenever G is strongly regular.

A weaker notion is that of an edge-regular graph. An *edge-regular graph* [7] with parameters (n, k, λ) is a graph on n vertices which is regular of degree k and such that any two adjacent vertices have exactly λ common neighbours. It is evident that strongly regular graphs are edge-regular, but the converse does not hold. In fact, the class of strongly regular graphs is far more restricted than that of edge-regular ones, with a rich structure that has attracted numerous researchers. For excellent overviews on the topic, see [10, 11, 17].

In the next section we give some necessary preliminaries on strongly regular graphs, and the subsequent section concerns the Gallai and anti-Gallai graphs of strongly regular graphs. All graph theoretic notations and terminology not defined here are standard, and can be found for instance in [5, 13].

2. Preliminaries on Strongly Regular Graphs

Henceforth, we assume G is a connected strongly regular graph with parameters (n, k, λ, μ). We will in many cases refer to subgraphs of G, and we must be careful since there are two different types of subgraphs which we will use. The first is an *induced subgraph*, which is formed from a subset of the vertices of G together with all edges connecting vertices in that set. The more general definition of subgraphs allows for edges from G to be absent even when both endpoints are present in the subgraph. To avoid confusion, we will always refer to induced subgraphs as such, and if we simply consider a subgraph (without the word "induced" present) then it is not assumed to be induced.

The following lemmas contain facts that are certainly known, but the proof is included for the benefit of readers unfamiliar with the structure of strongly regular graphs and standard methods of proof in the field.

Lemma 2.1. *Let G be a connected strongly regular graph with parameters (n, k, λ, μ).*

(1) $\mu \geq 1$.

(2) *If $\mu > 1$ then every pair of non-adjacent vertices must belong to at least one (not necessarily induced) cycle of length 4.*

(3) *Given an edge $e = uv$ and a vertex w, either there is a vertex y adjacent to all of w, u, v or else w, u, v must belong to a cycle C_n of length at most 5.*

(4) *Given a pair of edges uv and xy, either there is a vertex z adjacent to all of u, v, x, y, or else both edges lie on a cycle C_n of length at most 6.*

(5) *If $\mu = 1$ then diamonds and C_4's are forbidden as induced subgraphs in G (a diamond refers to two triangles sharing a common edge, i.e K_4 with an edge removed [23]).*

(6) *A strongly regular graph is 2-connected (that is, the removal of any vertex does not disconnect the graph).*

(7) *If $\lambda \leq 1$ then diamonds and K_4's are forbidden in G.*

Proof. We will employ the following notation, which is not entirely standard in the field. Given vertex x, let $N_j(x) = \{y : d(x, y) = j\}$. More standard would be to

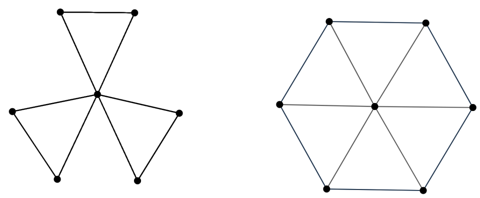

Figure 1: The friendship graph F_3 and wheel graph W_6

use Γ instead of N, but we will reserve Γ for the Gallai graph. Then, for any x, the vertices of G can be partitioned into the sets $\{x\}, N_1(x)$, and $N_2(x)$.

(1) This is trivial, since G is connected and of diameter 2.

(2) If $y \in N_2(x)$ and $\mu > 1$ then there are distinct $u, v \in N_1(x) \cap N_1(y)$ and then $xuyv$ is a cycle of length 4. Note that the triangular graph $T(4)$, which can also be realized as the complement of a perfect matching on 6 vertices, shows that this need not be an induced cycle.

(3) Since G has diameter 2 there are paths of length at most 2 from w to u and from w to v. If these paths do not share vertices other than w then they, together with uv, form a cycle of length at most 5. It is also possible, however, that the paths are of the form wxu and wxv for some vertex x, but then x is adjacent to w, u, v, and the conclusion holds.

(4) Follows similarly, since we can find paths of length at most 2 between u and x and between v and y, and then these paths can then either be used to form a cycle of length at most 6 or else yield a point adjacent to all of u, x, v, y.

(5) These graphs cannot exist as induced subgraphs since both contain vertices of distance 2 from each other but with two paths of length 2 connecting them.

(6) If we remove a vertex w from G, then we must show that the subgraph induced on $N_1(w) \cup N_2(w)$ is connected. Choose $x \in N_2(w)$, and let $y \in N_1(w) \cup N_2(w)$. Since the diameter of G is 2, there must be a path of length at most 2 from x to y, and this path can't pass through w since $d(w, x) = 2$. The result follows from this.

(7) Note that $\lambda \leq 1$ implies that any adjacent points can have at most one common neighbor. Both of the graphs indicated contain adjacent points with 2 common neighbors, and therefore can't exist as subgraphs of G. $\square$

Remark 2.2. In reference to (6) in Lemma 2.1, much more is known. A strongly regular graph is in fact k-connected: the removal of any $k-1$ vertices does not disconnect the graph, and in fact the only set of k vertices whose removal does disconnect the graph is the neighborhood of a vertex. This is non-trivial, though; see [9].

The *friendship graph*, denoted by F_n, is n copies of K_3 conjoined at a common vertex. Note that F_n is a planar graph with $2n+1$ vertices and $3n$ edges [20]. The following result gives a local characterization of strongly regular graphs with $\lambda = 1$.

Lemma 2.3. *Let G be a connected strongly regular graph with parameters $(n, k, 1, \mu)$. Then every vertex together with its neighbours form a friendship graph with $k/2 + 1$ vertices.*

Proof. Note that the handshake lemma implies k is even. Consider an arbitrary vertex u in G. Since G is connected, there exists an edge incident to u and by hypothesis that edge belongs to a K_3. $G \neq K_3$ and regular implies that there exists another edge incident to u but disjoint from the first triangle. But by *7* of the previous lemma, diamonds are forbidden in G and hence that edge also belongs to another K_3. Proceeding like this we find that u together with its neighbours form a friendship graph with $k/2 + 1$ vertices. $\square$

A *wheel graph* W_n is formed by joining a single vertex to all vertices of the n-cycle C_n. The following result indicates a structural characteristic of strongly regular graphs with $\lambda \geq 2$.

Lemma 2.4. *Let G be a connected strongly graph with parameters (n, k, λ, μ) with $\lambda \geq 2$. Then every vertex is a central vertex of at least one wheel.*

Proof. Since $\lambda \geq 2$, given an edge e the endpoints have at least 2 common neighbors, and e together with these two paths of length 2 form a diamond. Consider $u \in V(G)$, and let e_1 be any edge incident to u. e_1 is then the central edge of a diamond. Let e_2 and e_3 be the other edges of the diamond which are incident to u. Then there exist diamonds with e_2 and e_3 as their central edges. We proceed like this, and the process will terminate only when u together with some set of edges form a wheel. It may happen that there exists another edge incident to u which is not in the above wheel. In this case by the same argument we can find another wheel with u as the central vertex. $\square$

3. Gallai and Anti-Gallai Graphs of Strongly Regular Graph

The following indicates the advantages enjoyed by requiring graphs to be edge-regular (or strongly regular) rather than just regular. We reiterate that this result is false for regular graphs which are not edge-regular.

Theorem 3.1. *If G is an edge-regular graph with parameters (n, k, λ), then $\Gamma(G)$ and $\Delta(G)$ are regular graphs.*

Proof. Consider an arbitrary vertex x of $\Gamma(G)$. Let uv be the edge corresponding to this vertex in G. Then by the definition of $\Gamma(G)$, the degree of x is $d_{\Gamma(G)}(x) = d_G(u) + d_G(v) - 2|N_1(u) \cap N_1(v)| - 2$, where $d_{\Gamma(G)}(x)$ denotes the degree of x in $\Gamma(G)$ and $d_G(u), d_G(v)$ denote the degrees of u, v in G. Hence $\Gamma(G)$ is $2(k - \lambda - 1)$-regular.

Now consider an arbitrary vertex x in $\Delta(G)$. Let $e = uv$ be the corresponding edge in G. Then the degree of x in $\Delta(G)$ is $2|N_1(u) \cap N_1(v)| = 2\lambda$. Therefore, $\Delta(G)$ is 2λ-regular. □

We are interested in even stronger statements. In particular, we are interested in conditions under which the Gallai and anti-Gallai graphs are edge-regular, or even strongly regular. We will consider each of these separately in the following two subsections.

3.1. Gallai graphs

As per Theorem 3.1 we have that if G is strongly regular then $\Gamma(G)$ is regular. In this section we prove that the Gallai graphs of some special classes of strongly regular graphs are edge-regular or strongly regular. We begin with a more general structure theorem.

Theorem 3.2. *Let G be a connected graph. The Gallai graph $\Gamma(G)$ is disconnected if and only if there exists a partitioning of the edge set $E(G)$ into sets $E_1, E_2, ...E_p$ (where $p \geq 2$) such that $e_i \in E_i$ and $e_j \in E_j$ being incident in G implies that e_i and e_j span a triangle in G.*

Proof. Suppose that $\Gamma(G)$ is disconnected and let $\Gamma_1, \Gamma_2, ...\Gamma_p$ with $p \geq 2$ be the components of $\Gamma(G)$. Let E_i=$\{e \in G : e$ is an edge corresponding to a vertex v in $\Gamma_i\}$, where $1 \leq i \leq p$. Clearly, E_i is a partition for $E(G)$. Since the connectedness of G implies the connectedness of $L(G)$, at least one edge of E_i is incident with some $e \in E_j$ for some $j \neq i$. But, Γ_i and Γ_j are different components of $\Gamma(G)$, and hence if $e_i \in E_i$ is incident with $e_j \in E_j$ then they must span a triangle in G.

For the converse assume that such a partition exists for $E(G)$. Then, for any distinct i and j, the vertices corresponding to the edges in E_i and E_j are in different components in $\Gamma(G)$. □

We can say more in special cases. For instance, for $\lambda = 0$ we have the following.

Theorem 3.3. *Let G be a connected strongly regular graph with parameters $(n, k, 0, \mu)$. Then $\Gamma(G)$ is connected and edge-regular. Also it is strongly regular if and only if the following conditions hold:*

(1) *If $\mu = 1$ then any two non-adjacent edges belong to a common C_5.*

(2) *If $\mu > 1$ then any two non-adjacent edges belong to a common C_4.*

Proof. Since $\lambda = 0$, G is K_3-free and thus $\Gamma(G) \cong L(G)$. Since G is connected and k-regular, $L(G)$ is connected and $2k - 2$ regular, and thus so is $\Gamma(G)$.

Consider two adjacent vertices x and y in $L(G)$, and let e_1 and e_2 be the corresponding edges in G. Then e_1 and e_2 must have a common vertex in G. The number of common vertices of x and y in $L(G)$ is same as the number of edges incident on the common vertex of both e_1 and e_2. Since the graph is k-regular it is equal to $k-2$. Hence $\Gamma(G)$ is an edge-regular graph with parameters $(\frac{nk}{2}, 2k-2, k-2)$.

Now consider two non-adjacent vertices x and y in $L(G)$. Let f_1 and f_2 be the corresponding edges in G. Then f_1 and f_2 have no common vertex in G. The number of common vertices of x and y in $L(G)$ is same as the number of edges adjacent to both f_1 and f_2.

If $\mu = 1$, by Lemma 2.1 part 4, f_1 and f_2 belong to a cycle of length at most 6 (the other possibility, that one point is adjacent to all endpoints of f_1 and f_2, is not possible since $\lambda = 0$). The cycle cannot be of length 3 (because $\lambda = 0$) or 4 (because $\mu = 1$). If the cycle is of length 5, then the number of common vertices of x and y is 1, otherwise it is 0. Since G is strongly regular by Lemma 2.1 part *3* there exist non adjacent edges which belong to a C_5. So $L(G)$ is strongly regular if and only if any two non-adjacent edges belong to a C_5.

We should note that if $\mu = 1$ then G is a Moore graph, and there is a well-known classification of such graphs (see [17]). G is therefore C_5, the Petersen graph, the Hoffman-Singleton graph, or the famous hypothetical Moore graph of valency 57. Of these, C_5 is the only one such that any two non-adjacent edges belong to a common C_5.

The same reasoning applies if $\mu > 1$, except that now f_1 and f_2 may belong to C_4, C_5 or C_6. If f_1 and f_2 belong to C_4 then the number of common vertices of x and y is 2; otherwise it is 1 or 0. Since $\mu > 1$, in G there are edges which belong to C_4. So $\Gamma(G)$ is strongly regular if and only if any two non-adjacent edges belong to a C_4. □

We also have the following result when $\lambda = 1$.

Theorem 3.4. *If G is a connected strongly regular graph with parameters $(n, k, 1, \mu)$, then $\Gamma(G)$ is edge-regular but not strongly regular.*

Proof. Since G is strongly regular, $\Gamma(G)$ is $2(k-2)$-regular by Theorem 3.1. Now consider two adjacent vertices v_1 and v_2 in $\Gamma(G)$. If e_1 and e_2 be the corresponding edges in G, then the number of common vertices of v_1 and v_2 is same as the number of induced $K_{1,3}$'s in which e_1 and e_2 are present. By Lemma 2.3, since the number of such induced $K_{1,3}$ is $k-2$, any two adjacent vertices have $k-2$ common neighbours. Hence $\Gamma(G)$ is an edge-regular graph with parameters $(\frac{nk}{2}, 2k-4, k-2)$.

To prove that $\Gamma(G)$ is strongly regular, consider two vertices v_1 and v_2 which are non-adjacent in $\Gamma(G)$. Let e_1 and e_2 be the corresponding edges in G. Then in G either e_1 and e_2 span a triangle or e_1 and e_2 are non-adjacent.

In the first case, the number of common neighbours of v_1 and v_2 is same as the number of edges which form $K_{1,2}$ with both e_1 and e_2. By Lemma 2.3 it is same as $k-2$.

In the latter case, the number of common vertices is same as the number of induced P_4's with end edges e_1 and e_2. To find the number of induced P_4's we consider the following cases.
1) $\mu = 1$
2) $\mu > 1$
If $\mu = 1$ by Observation 3, e_1 and e_2 may belong to a C_5 or C_6. By Observation 5, C_4's are forbidden in G. Therefore the number of such induced P_4 is one if e_1 and e_2 belong to a C_5; otherwise it is zero. Since G contains C_5, $\Gamma(G)$ is strongly regular if and only if $k-2=1$ and any two non-adjacent edges belong to at least one C_5. But this is not possible since k is an even number by Lemma 2.3.

If $\mu > 1$, by the same argument in the above theorem $\Gamma(G)$ is strongly regular if and only if $k-2=2$ and any two edges belong in a C_4. Hence we conclude that $\Gamma(G)$ is strongly regular if and only if $k=4, \mu > 1$ and any two non-adjacent edges belong to a C_4. However, the strongly regular graphs of valency 4 are known, and can be found in [8]. They are the octahedron (with $\lambda = 2$), the complete bitartite graph $K_{3,3}$ (with $\lambda = 0$), and the 3×3 grid (which has non-adjacent edges not belonging to a C_4, as may be easily checked). Therefore there are no strongly regular graphs satisfying this requirement, and the theorem follows. □

Furthermore, we can deduce a result on the connectivity of $\Gamma(G)$ under the same conditions as for the previous two results.

Theorem 3.5. *Let G be a connected strongly regular graph with parameters (n, k, λ, μ). Suppose that $\lambda = 0$ or 1. Then $\Gamma(G)$ is 2-connected.*

Proof. We have the following cases.

Case 1: $\lambda = 0$
Since G is K_3-free, $\Gamma(G) \cong L(G)$. When we consider any two vertices of $L(G)$, by Lemma 2.1 Part *4*, the corresponding edges belong to a C_4, C_5 or C_6 (there can't be a vertex adjacent to all endpoints of the corresponding edges, since G is K_3-free). So in $L(G)$ any two vertices belong to a C_4, C_5 or C_6. Hence it is 2-connected.

Case 2: $\lambda = 1$
In order to show that $\Gamma(G)$ is 2-connected it is enough to show that any two vertices in $\Gamma(G)$ belong to a cycle. Consider two vertices x and x' in $\Gamma(G)$. We consider separately the following cases.
(1) xx' is an edge in $\Gamma(G)$.
(2) xx' is not an edge in $\Gamma(G)$.

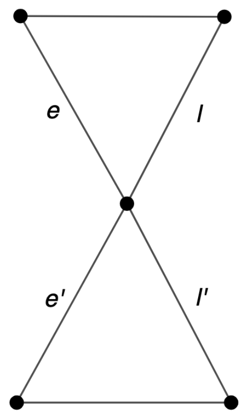

Figure 2: Edges forming a C_4 in $\Gamma(G)$

Subcase A: xx' is an edge in $\Gamma(G)$ implies that the corresponding edges e and e' are incident in G and do not belong to a K_3. By Lemma 2.3 there exists an edge l such that e and l span a K_3 in G. Similarly there exists edge l' such that e' and l' span a K_3 in G. Then the vertices corresponding to e, l', l, e' form a path P_4 in $\Gamma(G)$. Then this path together with the edge xx' form a C_4 in $\Gamma(G)$.

Figure 2 shows the edges used in this argument to form a cycle in $\Gamma(G)$. The reader may wish to take the time to thoroughly digest this argument, as variants of it will be used throughout the next subcase.

Subcase B: xx' not an edge in $\Gamma(G)$, which means that either the corresponding edges are incident in G and belong to a K_3 or are not incident. For the first case, by Lemma 2.3, there exist at-least two edges l and l' which span a K_3 in G. Then the vertices corresponding to e, l, e', l', e span a C_4 in $\Gamma(G)$. For the latter case by Lemma 2.1 part *4*, either e and e' belong to a cycle C_n where $3 < n < 7$ in G, or there is a vertex adjacent to all endpoints of e, e'. Let us consider the cycle case first. If it is an induced C_n then clearly x and x' belong to a C_n in $\Gamma(G)$. If not, there are edges which belong to a K_3 in G. Consider two edges e_1 and e_2 with a common vertex u and span a K_3 in G. By Lemma 2.3 there exist another edges l and l' incident on u which span a K_3 in G. Since diamonds are forbidden in G, the vertices corresponding to e_1, l, e_2 form a P_3 in G. So, if two edges span a K_3 in G then by the above explanation we can find an edge such that the vertices corresponding to the edges of K_3 and the new edge form a P_3 in G. Also if e_1, e_2, e_3 are three consecutive edges in C_n and if e_1 and e_2 span a K_3 in G, since diamonds are forbidden in G, e_2 and e_3 cannot span a K_3 in G. So by the above explanation in $\Gamma(G)$, we can find a cycle C_n of length at most 9 containing the vertices x and x'. That is, in $\Gamma(G)$ any two vertices belong to at least one C_n. Hence $\Gamma(G)$ is 2-connected.

Finally, if there is a vertex v adjacent to each endpoint of e and e' then a similar argument allows us to build a cycle in $\Gamma(G)$, as follows. Since $k \geq 4$ each endpoint of e is adjacent to a point other than v, and this can't be an endpoint of e' since diamonds are forbidden. This new edge cannot span a triangle with e or containing v, since $\lambda = 1$. Similar new edges can be found at the endpoints of e'. The edges $ef_1s_1s_2'f_2'e'f_1's_1's_2f_2e$ form a C_{10} in $\Gamma(G)$. □

3.2. Anti-Gallai Graphs

The following is the analogue of Theorem 3.2 for anti-Gallai graphs.

Theorem 3.6. *Let G be a connected graph. The anti-Gallai graph $\Delta(G)$ is disconnected if and only if there exists a partition of the edge set of G into $E_1, E_2, ...E_p$ where $p \geq 2$, such that if $e_i \in E_i$ and $e_j \in E_j$ are incident in G then e_i and e_j do not belong to a K_3, and there exists at least one pair of this type.*

Proof. Suppose that $\Delta(G)$ is disconnected and let $\Delta_1, \Delta_2, ...\Delta_p$ with $p \geq 2$ be the components of $\Delta(G)$. As in the proof of Theorem 3.2 consider

$$E_i = \{e \in G : e \text{ is an edge corresponding to a vertex } v \text{ in } \Delta_i\},$$

where $1 \leq i \leq p$. Clearly, E_i is a partition for $E(G)$. Since the connectedness of G implies the connectedness of $\text{Ł}(G)$, at least one edge $e_i \in E_i$ is incident with some $e_j \in E_j$ with $j \neq i$. However, Δ_i and Δ_j are different components of $\Delta(G)$, and hence if $e_i \in E_i$ is incident with $e_j \in E_j$ then they do not belong to a triangle in G.

For the converse assume that such a partition exists for $E(G)$. Then for any i and j, the vertices corresponding to the edges in E_i and E_j induce different components in $\Delta(G)$. □

We can deduce from this a result on strongly regular graphs with $\lambda = 2$ which are locally cycles, which are graphs such that the subgraph induced on $N_1(x)$ for any x is a cycle. A prominent example of a graph of this type is the Shirkhande graph, which has parameters $(16, 6, 2, 2)$. The result is as follows.

Theorem 3.7. *Let G be a connected strongly regular graph with parameters $(n, k, 2, \mu)$ which is locally a cycle. Then $\Delta(G)$ is connected and edge-regular.*

Proof. Suppose $\Delta(G)$ is disconnected. Then by Theorem 3.6 there exists at least two edges $e_i \in E_i$ and $e_j \in E_j$ which are incident but do not belong to a K_3 in G. Let u be the common vertex of both e_i and e_j. By assumption, since the neighbouring vertices induce a wheel in G, there exist edges $e_{i+1}, e_{i+2}, ..., e_{j-1}, e_j$ such that the pairs of edges $(e_i, e_{i+1}), (e_{i+1}, e_{i+2}), ..., (e_{j-1}, e_j)$ belongs to a K_3 in G. Since $e_i \in E_i$, the edges $e_{i+1}, e_{i+2}, ..., e_{j-1}, e_j$ all belong to E_i. This is a contradiction, and hence $\Delta(G)$ is connected.

By Theorem 3.1, $\Delta(G)$ is a 4-regular graph. Since G is not K_4 any two adjacent vertices in $\Delta(G)$ have only one common neighbour. Hence $\Delta(G)$ is edge-regular with parameters $(\frac{nk}{2}, 4, 1)$. □

The following is a series of examples illustrating our results.

Example 3.8. Let G be a connected strongly regular graph with parameters $(n, k, 0, \mu)$. Since $\lambda = 0$, G is K_3-free. Hence $\Delta(G)$ is totally disconnected. The spectrum of $\Delta(G)$ is $(0^{\frac{kn}{2}})$.

Example 3.9. Let G be a connected strongly regular graph with parameters $(n, k, 1, \mu)$. Since $\lambda = 1$, every edge of G belong to exactly one K_3 and no two K_3 share a common edge. Therefore $\Delta(G)$ is the disjoint union of $\dfrac{kn}{6}$ triangles. $\Delta(G)$ therefore has the spectrum $(-1^{\frac{kn}{3}}, 2^{\frac{kn}{6}})$.

Example 3.10. Let us consider the anti-Gallai graph of the line graph of the complete bipartite graph, $L(K_{n,n})$. $L(K_{n,n})$ contains $2n$ copies of K_n sharing common vertices, where two K_n's have no common edges and the edges of two copies of K_n do not belong to a K_3. Hence $\Delta(L(K_{n,n}))$ is the disjoint union of $2n$ copies of $\Delta(K_n)$. Since every pair of edges in K_n spans a triangle, $\Delta(K_n) \cong L(K_n)$, and $L(K_n)$ is a well-known family of graphs known as the *triangular graphs*. These are strongly regular with parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$ and spectrum $(2(n-2)^1, (n-4)^{n-1}, -2^{n(n-3)/2})$. Hence the spectrum of $\Delta(L(K_{n,n}))$ is $(2(n-2)^{2n}, (n-4)^{2n(n-1)}, -2^{n^2(n-3)})$.

Our final result applies to a situation in which we are not able to identify $\Delta(G)$ easily but in which we can say something about the spectrum only.

Definition 3.11. Suppose $\alpha_1 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \ldots \leq \beta_m$, with $m < n$. We will say the sequence of β's *interlace* the sequence of α's if $\alpha_k \leq \beta_k \leq \alpha_{k+n-m}$ for $k = 1, \ldots, m$.

The following two lemmas are famous results in spectral graph theory.

Lemma 3.12. [12] *Let G be a graph and H an induced subgraph of G. Then the eigenvalues of H interlace those of G.*

Lemma 3.13. [5] *If G is a k-regular graph, then $spec(G) \in [-k, k]$, and $k \in spec(G)$.*

We need two further definitions.

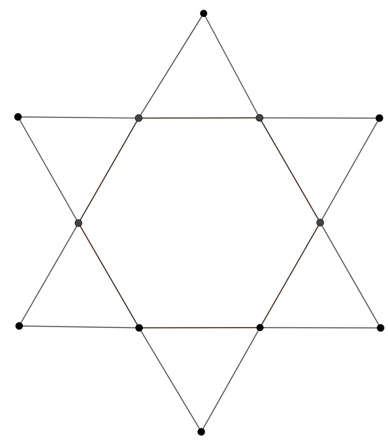

Figure 3: $R(C_6)$

Definition 3.14. Let G_1 and G_2 be vertex-disjoint graphs. The *join* of G_1 and G_2, denoted $G_1 \vee G_2$, is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2 [5].

Definition 3.15. The *semi-total point graph* $R(G)$ of a graph G is obtained from G by adding a new vertex corresponding to every edge of G, then joining each new vertex to the end vertices of the corresponding edge i.e; each edge of G is replaced by a triangle [28].

The semi-total point graph of a cycle, $R(C_n)$, will be of special interest to us, and Figure 3 represents this graph for $n = 6$. The following lemma connects joins, semi-total point graphs, and anti-Gallai graphs.

Lemma 3.16. [24] *If $G = H \vee K_1$, where H is K_3-free, then $\Delta(G)$ is the semi-total point graph of H.*

We are now prepared to prove the following theorem.

Theorem 3.17. *Let G be a connected strongly regular graph with parameters $(n, k, 2, \mu)$ which is locally a cycle. Then spec$(\Delta(G)) \subseteq [-4, 4]$ and the eigenvalues of $R(C_k)$ interlace those of $\Delta(G)$.*

Proof. By Theorem 3.7, $\Delta(G)$ is a 4-regular graph. Therefore, by Lemma 3.13 [5] the largest eigenvalue of $\Delta(G)$ is 4 and *spec*$(G) \in [-4, 4]$. Also, by assumption each vertex belong to exactly one wheel $(C_n \vee K_1)$. Since $\lambda = 2$, a vertex in G together with its neighbours form an induced k-wheel. Then, by Lemma 3.16 [24], it is clear that $\Delta(G)$ contains the semi-total point graph $R(C_k)$ as an induced subgraph. Therefore, by Lemma 3.12 [12], the eigenvalues of $R(C_k)$ interlace those of $\Delta(G)$. □

To apply this result requires knowledge of the spectrum of $R(C_k)$. The eigenvectors of this graph can be found in a similar manner to that of a cycle, namely by placing powers of a complex n-th root of unity at the points of the cycle, and then finding what the remaining values must be at the other points. We omit the details, but the reader may check that the spectrum of $R(C_k)$ is the values of the form $\frac{r\pm\sqrt{r^2+4r+8}}{2}$, where $r = 2\cos(2\pi k/n)$ for $k = 1, \dots, n$.

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