TOPOLOGICALLY FREE ACTIONS AND PURELY INFINITE $C^*$-CROSSED PRODUCTS

JA A JEONG

1. Introduction

For a given $C^*$-dynamical system $(A, G, \alpha)$ with a $G$-simple $C^*$-algebra $A$ (that is $A$ has no proper $\alpha$-invariant ideal) many authors have studied the simplicity of a $C^*$-crossed product $A \rtimes_{\alpha} G$. In [1] topological freeness of an action is shown to guarantee the simplicity of the reduced $C^*$-crossed product $A \rtimes_{\alpha r} G$ when $A$ is $G$-simple.

In this paper we investigate the pure infiniteness of a simple $C^*$-crossed product $A \rtimes_{\alpha} G$ of a purely infinite simple $C^*$-algebra $A$ and a topologically free action $\alpha$ of a finite group $G$, and find a sufficient condition in terms of the action on the spectrum of the multiplier algebra $M(A)$ of $A$. Showing this we also prove that some extension of a topologically free action is still topologically free.

2. Topologically free action

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Then there is an action of $G$ on the spectrum $\hat{A}$ of $A$; for each $\pi \in \hat{A}$, $t\pi(a) = \pi(\alpha_t(a))$, $t \in G, a \in A$. An action $\alpha$ is said to be topologically free if for any $t_1, \ldots, t_n \in G \setminus \{e\}$ the set $\bigcap_{i=1}^n \{ \pi \in \hat{A} \mid t_i \pi \neq \pi \}$ is dense in $\hat{A}$.

Remark 1. If $A$ is simple, then $\hat{A}$ is the only nonempty open set of $\hat{A}$. This is because each open set in $\hat{A}$ corresponds to an ideal of $A$. Hence, for $A, \alpha$ is topologically free if and only if $\bigcap_{i=1}^n \{ \pi \in \hat{A} \mid t_i \pi \neq \pi \}$ is nonempty for every $t_1, \ldots, t_n \in G \setminus \{e\}$.

1991 Mathematics Subject Classification. 46L05, 46L55.
Supported by Global Analysis Research Center.
In [1], it was shown that if $\alpha$ is topologically free, then each automorphism $\alpha_t$ is properly outer for $t \setminus \{e\}$; that is, for every nonzero $\alpha_t$-invariant ideal $I$ of $A$ and inner automorphism $\beta$ of $I$, $\| \alpha_t|_I - \beta \| = 2$. Hence, if $\alpha$ is a topologically free action on a simple $C^*$-algebra $A$ by a group $G$, then each automorphism $\alpha_t$ is outer for $t \setminus \{e\}$. Conversely, as it was mentioned in [1], if $A$ is a separable simple $C^*$-algebra, and each $\alpha_t(t \neq e)$ is outer, then $\alpha$ is topologically free. This follows because topological freeness is weaker than the strong Connes spectrum condition used in [8], and the spectrum condition is equivalent to the outerness of $\alpha$ for simple $C^*$-algebras.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system with a discrete group $G$. Then the action $\alpha$ uniquely extends to an action on the multiplier algebra $M(A)$ of $A$, and we write this extension by $\alpha$ again.

Recall that a $C^*$-algebra $A$ is said to be purely infinite if every hereditary $C^*$-subalgebra of $A$ has an infinite projection, a projection equivalent to its subprojection. Obviously, every hereditary $C^*$-subalgebra of a purely infinite $C^*$-algebra is purely infinite. It is not known whether a simple $C^*$-algebra containing an infinite projection is purely infinite or not. For properties and examples of purely infinite $C^*$-algebras, refer to [3], [12], and [13].

If $A$ is purely infinite, then so is $M(A)$, and the following is proved by Rørdam:

**Proposition 2.** [12] Let $A$ be a unital simple $C^*$-algebra, and let $\mathcal{K}$ denote the $C^*$-algebra of compact operators on an infinite dimensional separable Hilbert space. Then $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple if and only if $A$ is the matrix algebra $M_n(\mathbb{C})$ or purely infinite.

**Proposition 3.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $A$ is a $\sigma$-unital nonunital purely infinite simple $C^*$-algebra. If $\alpha$ is topologically free, then so is the extension $\alpha$ on $M(A)$.

**Proof.** For $t_1, \ldots , t_n \in G \setminus \{e\}$, the set $X = \bigcap_{i=1}^n \{ \pi \in \hat{A} | t_i \pi \neq \pi \}$ is dense in $\hat{A}$. Each representation $\pi \in X$ on a Hilbert space $H$ extends uniquely to $M(A)$ [4, Proposition 2.10.4] on $H$ so that the extension, also denoted by $\pi$, is still irreducible.

Let $T$ be a bounded operator on $H$ intertwining $\pi$ and $t_i \pi$; that is,
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\[ T\pi(x) = t_i\pi(x)T \text{ for } x \in M(A). \] Then \( T \) automatically intertwines \( \pi \) and \( t_i\pi \) on \( A \). Hence, \( T \equiv 0 \) because \( \pi \) and \( t_i\pi \) are disjoint [9, Corollary 3.13.3]. Therefore, \( \pi \) and \( t_i\pi \) are disjoint as representations of \( M(A) \) so that \( Y = \bigcap_{i=1}^{n} \{ \pi \in M(A)^\sim | t_i\pi \neq \pi \} \) has nonempty intersection with \( \hat{A} \). Since \( A \) is nonunital, \( A \) is stable [14, Theorem 1.2] and we hence have \( A \cong A \otimes K \cong A_0 \otimes K \) for each unital hereditary (simple) C*-subalgebra \( A_0 \) of \( A \) since a \( \sigma \)-unital simple C*-algebra is stably isomorphic to its hereditary C*-subalgebra [2, Corollary 2.6]. Therefore, \( A \) is the unique ideal in \( M(A) \) by Proposition 2; that is, \( \hat{A} \) is the unique proper open subset in \( M(A)^\sim \), and we conclude that \( Y \) is dense in \( M(A)^\wedge \), and we conclude that \( Y \) is dense in \( M(A)^\wedge \).

Given a C*-dynamical system \((A, G, \alpha)\), we have an induced C*-dynamical system \((A \otimes K, \tilde{\alpha} = \alpha \otimes \text{id}, G)\). For a unital simple C*-algebra \( A \), we will show that \( \tilde{\alpha} \) is topologically free if \( \alpha \) is. If \( \pi : A \to B(H) \) is an irreducible representation of \( A \), then \( \tilde{\pi} = \pi \otimes \text{id} : A \otimes K \to B(H \otimes H') \) is still irreducible [7, Proposition 11.3.2], where \( H' \) is a separable infinite dimensional Hilbert space. Note that given a C*-dynamical system \((A, G, \alpha)\), we have the action of \( G \) on the spectrum \((A \otimes K)^\wedge \) given by \( \hat{t}\tilde{\pi}(x) = \tilde{\pi}(\tilde{\alpha}_t(x)) \) for \( x \in A \otimes K, t \in G \), and \( \tilde{\pi} \in (A \otimes K)^\wedge \).

**Lemma 4.** If \( \pi : A \to B(H) \) is an irreducible representation of a unital C*-algebra \( A \) such that \( t\pi \neq \pi \), then \( \hat{t}\tilde{\pi} \neq \tilde{\pi}, t \in G \setminus \{e\} \).

**Proof.** We show that \( \hat{t}\tilde{\pi} \) is disjoint with \( \tilde{\pi} \). Let \( T \) be an intertwining operator of \( \hat{t}\tilde{\pi} \) and \( \tilde{\pi} \) on \( H \otimes H' \), that is, \( \tilde{\pi}(x)T = T\hat{t}\tilde{\pi}(x) \) for \( x \in A \otimes K \).

For a nonzero vector \( \xi_0 \in H' \), let \( p_0 \in K \) be the projection onto the one dimensional subspace \( \langle \xi_0 \rangle \) of \( H' \) so that \( 1 \otimes p_0 \in A \otimes K \).

For each vector \( \eta \otimes \xi_0 \in H \otimes \xi_0 \), we have

\[
(1 \otimes p_0)T(\eta \otimes \xi_0) = \tilde{\pi}(1 \otimes p_0)T(\eta \otimes \xi_0)
\]
\[
= T\hat{t}\tilde{\pi}(1 \otimes p_0)(\eta \otimes \xi_0)
\]
\[
= T\tilde{\pi}(\alpha_t(1) \otimes p_0)(\eta \otimes \xi_0)
\]
\[
= T(\eta \otimes \xi_0).
\]
Let \( \{\xi_i\}_{i=0}^{\infty} \) be an orthogonal basis of \( H' \). Then we can write

\[
T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0 + \sum_{i=1}^{\infty} \eta_i \otimes \xi_i
\]

for some \( \eta_i \in H \) and \((1 \otimes p_0)T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0\). Hence, \( T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0 \) for some \( \eta_0 \in H \). Moreover, it is not difficult to show that the map \( T_0 : H \to H \) defined by \( T_0(\eta) = \eta_0 \) is bounded linear. For \( a \otimes p_0 \in A \otimes \mathcal{K}, a \in A \), we have

\[
\pi(a)\eta_0 \otimes \xi_0 = (\pi(a) \otimes p_0)(\eta_0 \otimes \xi_0) = (\pi(a) \otimes p_0)(T(\eta \otimes \xi_0))
\]

\[
= \tilde{\pi}(a \otimes p_0)T(\eta \otimes \xi_0) = T\tilde{\pi}(a \otimes p_0)(\eta \otimes \xi_0)
\]

\[
= T(\pi(\alpha_t(a)) \otimes p_0)(\eta \otimes \xi_0)
\]

\[
= T(\pi(\alpha_t(a))\eta \otimes \xi_0) = (\pi(\alpha_t(a))\eta)_0 \otimes \xi_0
\]

Hence, \( \pi(a)\eta_0 = (\pi(\alpha_t(a))\eta)_0 \); that is, \( \pi(a)T_0 = T_0(t\pi(a)) \) for every \( a \in A \) because the map \( H \to H \otimes \xi_0 \) given by \( \eta \mapsto \eta \otimes \xi_0 \) is injective. Since \( \pi \) and \( t\pi \) are disjoint, we conclude that \( T_0 \equiv 0 \). Therefore, \( T \equiv 0 \) because \( \xi_0 \) is an arbitrary non zero vector of \( H' \).

**Theorem 5.** Let \((A \otimes \mathcal{K}, G, \tilde{\alpha} = \alpha \otimes \text{id})\) be a \( C^* \)-dynamical system where \( A \) is a simple \( C^* \)-algebra. Then \( \tilde{\alpha} \) is topologically free if so is \( \alpha \).

**Proof.** Since \( A \otimes \mathcal{K} \) is simple, it suffices to show that for every \( t_1, \ldots, t_n \in G \setminus \{e\} \), the set \( \bigcap_{i=1}^{n} \{ \tilde{\pi} \in (A \otimes \mathcal{K})^k | t_i \tilde{\pi} \neq \tilde{\pi} \} \) is nonempty. But this is almost obvious by Lemma 4.

### 3. Purely infinite simple \( C^* \)-crossed products

Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system with a purely infinite simple \( C^* \)-algebra \( A \) and a finite group \( G \).

In this section, we examine the pure infiniteness of the infinite \( C^* \)-algebra \( A \rtimes_{\alpha} G \).
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**Theorem 6.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system with a purely infinite simple $C^*$-algebra $A$ and a finite group $G$. If $\alpha$ is a topologically free action such that

$$\bigcap_{t \in G \setminus \{e\}} \{ \bar{\pi} \in (M(A \otimes \mathcal{K}))^\wedge | \tilde{t}\bar{\pi} \neq \bar{\pi} \} \cap \hat{A}^G \neq \emptyset,$$

then the infinite simple $C^*$-crossed product $A \rtimes_\alpha G$ is purely infinite.

**Proof.** It is known that the fixed point algebra $A^\alpha$ can be regarded as a hereditary $C^*$-subalgebra of the infinite simple $C^*$-algebra $A \rtimes_\alpha G$ whenever $G$ is compact [10]; hence, $A^\alpha$ contains a projection $p$ [5]. The unital hereditary $C^*$-subalgebra $A_p$ of $A$ generated by $p$ is invariant under $\alpha$. Moreover, from [11, Lemma 3.4], we see that if $A_p \rtimes_\alpha G$ is purely infinite, then so is $A \rtimes_\alpha G$. Actually, for each hereditary $C^*$-subalgebra $B$ of $A \rtimes_\alpha G$, we can find a unitary element $u \in M(A \rtimes_\alpha G)$ such that $uBu^* \cap (A_p \rtimes_\alpha G) \neq 0$; hence, $B$ has an infinite projection if $A_p \rtimes_\alpha G$ is purely infinite. Hence, we may assume that $A$ is unital. Note that $A \rtimes_\alpha G$ is purely infinite if and only if $(A \rtimes_\alpha G) \otimes \mathcal{K}$ is purely infinite [14, Proposition 1.4]. Since $(A \rtimes_\alpha G) \otimes \mathcal{K} \cong (A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G$ [6, Theorem 2.6] it suffices to show that $M((A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G)/(M(A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G)$ is simple by Proposition 2. Note that $M((A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G) = M(A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G$ ($G$ is finite) and

$$M((A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G)/(M(A \otimes \mathcal{K}) \rtimes_\bar{\alpha} G) \cong (M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \rtimes_\bar{\alpha} G.$$

The fact that $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple implies that the $C^*$-algebra $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \rtimes_\bar{\alpha} G$ is simple if $\bar{\alpha}$ is topologically free. Our assumption says that there is an irreducible representation $\bar{\pi}$ of $M(A \otimes \mathcal{K})$ with $\ker \bar{\pi} = A \otimes \mathcal{K}$ and $\tilde{t}\bar{\pi} \neq \bar{\pi}$ for $t \in G \setminus \{e\}$, which means that

$$\bigcap_{t \in G \setminus \{e\}} \{ \bar{\pi} \in (M(A \otimes \mathcal{K})/(A \otimes \mathcal{K}))^\wedge | \tilde{t}\bar{\pi} \neq \bar{\pi} \} \neq \emptyset,$$

and hence $\bar{\alpha}$ is topologically free by Remark 1.
Remark 7. An action $\alpha$ satisfying the condition in the above theorem induces an outer action of $G$ on a purely infinite simple $C^*$-algebra $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$. In case $\alpha$ induces an inner action of $G$ on $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ so that $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K}) \times _{\hat{\alpha}} M(A \otimes \mathcal{K})/(A \otimes \mathcal{K}) \otimes C^*(G)$ for the group $C^*$-algebra $C^*$ of $G$ then the crossed product $A \times _{\alpha} G$ is purely infinite whenever $C^*(G)$ is simple since $C^*(G)$ is just a matrix algebra.

References