Some Partial Orders Describing Positive Aging

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Abstract The concept of positive aging describes the adverse effects of age on the lifetime of units. Various aspects of this concept are described in terms of conditional probability distribution of residual life times, failure rates, equilibrium distributions, etc. In this paper we will consider some partial ordering relations of life distribution under residual life functions and equilibrium distributions.

Key Words: Life distribution, Residual Life Distribution, Partial Orderings, Equilibrium Distribution, Positive Ageing classes.

1. Introduction

By the aging of a mechanical unit, component, or some other physical or biological systems, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one.

Suppose that $X$ and $Y$ are nonnegative absolutely continuous random variables with probability density functions $f(x)$ and $g(x)$, respectively. Let $F$ and $G$ be the cumulative distribution functions of $X$ and $Y$, and $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x) = 1 - G(x)$ be the corresponding survival functions. Partial orderings, namely, $s-FR$ ordering (likelihood ratio ordering, failure rate ordering, mean residual life ordering), $s-ST$ ordering (weak likelihood ratio ordering, stochastic ordering, harmonic average mean life ordering), $s-SFR$ ordering (expectation ordering and initial failure rate ordering) between two random variables $X$ and $Y$ are known in the literature.

Deshpande, Kochar and Singh (1986) introduced various aspects of this

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concept which are described in terms of conditional probability distributions of residual life times, failure rates, equilibrium distributions. Gupta(1987) studied how the ageing properties IFR, NBU, NBUE and DMRL of the original distribution were transformed into the ageing properties of the distribution of the residual life. Kochar and Wiens(1987) defined new partial orderings of life distributions and studied the relationships of some partial orderings. Singh(1989) defined two new partial orderings and discussed relevance of these partial orderings for comparing life of a new unit with residual life of a used unit. Deshpande,Singh,Bagai and Jain(1990) investigated partial orders relations with existing partial orders of probability distributions for describing the phenomenon of ageing.


The residual life of a component of age and the equilibrium distribution are of great interest in actuarial studies, survival analysis and reliability. So, in Section 2, we shall consider some partial order relations of life distribution under the residual life function. In Section 3, we shall consider some partial order relations of life distribution under the equilibrium distribution.

2. Relations among Some Partial Orderings Under Residual Life Distributions

First of all, we briefly discuss below some of the well-known partial orderings(see Fagiuoli and Pellerey(1993)).

Given an absolutely continuous nonnegative random variable $X$, we denote $\overline{T}_0(x) = f(x)$ and $\overline{T}_s(x) = \int_x^{\infty} \frac{T_{s-1}(\mu)d\mu}{\mu_s}$, for $s \geq 1$, where $\mu_s(x) = \int_0^{\infty} \overline{T}_s(\mu)d\mu$.

Also define
Some Partial Orders

\[ r_{s}(x) = \frac{T_{s-1}(x)}{\int_{x} T_{s-1}(\mu) d\mu} = -\frac{d}{dx} T_{s}(x) \text{, for } s \geq 1 \]

and

\[ r_{s}(x) = \frac{f'(x)}{f(x)} \text{ when } f'(x) \text{ exists.} \]

We use \( \overline{U}_{s}(x) \), \( \overline{U}_{s}(x) \) and \( \gamma_{s} \) corresponding to \( T_{s}(x) \), \( r_{s}(x) \) and \( \mu_{s} \) for the random variables \( Y \).

**Definition 1.** \( X \) is said to be larger than \( Y \) in s-FR ordering, written as \( X \preceq_{s-\text{FR}} Y \), if \( \frac{T_{s}(x)}{U_{s}(x)} \) is nondecreasing in \( x \geq 0 \).

**Definition 2.** \( X \) is said to be larger than \( Y \) in s-ST ordering, written as \( X \preceq_{s-\text{ST}} Y \), if \( \frac{T_{s}(x)}{U_{s}(x)} \geq \frac{T_{s}(0)}{U_{s}(0)} \) for all \( x \geq 0 \).

**Definition 3.** \( X \) is said to be larger than \( Y \) in s-SFR ordering, written as \( X \preceq_{s-\text{SFR}} Y \), if \( r_{s}(0) \leq r_{s}(0) \).

**Definition 4.** \( X \) is said to be a s-IFR if \( r_{s}(x) \) is nondecreasing in \( x \geq 0 \).

**Theorem 2.1.** \( X \) be nonnegative s-IFR random variables, and let \( a \leq 1 \) be positive constant. Then \( aX \preceq_{s-\text{FR}} X \).

Proof. It is easy to verify that the functions \( r_{s}(x) \) of \( aX \) is given by

\[ \frac{1}{a} r_{s}(x/a) , \text{ for all } t, \text{ where } r_{s}(x) \text{ is the failure function of s-order equilibrium distribution of } X \].

Now,

\[ r_{s}(x) = \frac{1}{a} r_{s}(x/a) \geq r_{s}(x/a) \geq r_{s}(x) \text{, for all } x, \text{ where the first inequality follows from } a \in [0, 1] \text{ and the second inequality follows from the assumption that } X \text{ is s-IFR.} \]

Note that if \( X \) is the life time of a device then \( [X - t|X > t] \) is the residual life
of such a device with age $t$, and has the survival function 

$$F_t(x) = P[X > x + t | X > t] = F(x + t) / F(t).$$

**Lemma 2.2.** (Fagiuli and Pellerey(1993)) Let $X$ have distribution $F$ and $X_i$ have distribution $G$. With $T_s$ and $U_s$, we will mean the equilibrium distribution of order $s$ of $X$ and $X_i$, respectively.

$$U_s(x) = \frac{T_s(x + t)}{T_s(t)}, \text{ for all } s \in N^+, \text{ where } N^+ = \{x | x \text{ is natural number}\}.$$ 

We note that 

$$r_{U_s}(x) = \frac{-d}{dx} U_s(x) = \frac{-d}{dx} \frac{T_s(x + t)}{T_s(t)} = r_{T_s}(x + t).$$

Many partial orders are utilized for making comparisons between probability distributions of residual life times at different ages in order to describe positive ageing. Hence, we distinguish between two types of positive ageing.

i) Younger Better than Older (YBO) type ageing wherein the effect of ageing is progressive and the unit deteriorates monotonically, in some sense, with increasing age, i.e., the probability distributions of $X_i$, $t > 0$ are seen to be ordered monotonically in the sense of a suitable order.

ii) New Better than Used (NBU) type ageing wherein the comparison is only between a new unit (i.e., of age 0) and a used unit (i.e., of age $t > 0$). Positive ageing is asserted if the distributions of $X_i$, $t > 0$ can be ordered with respect to the distribution of $X$, the life time of a new unit. We observe that both the YBO and NBU comparisons in terms of some specific partial order lead to the same class of distributions.

We know that $X \geq X_i$, $t > 0$ if and only if $X$ is s-IFR.

**Theorem 2.3:** The following conditions are equivalent.

1. $X$ is s-IFR.
2. $X \geq X_i$, $t > 0$
3. \( X_{t_1}^{s-FR} \geq X_{t_2}^{s-FR} \), for all \( 0 < t_1 \leq t_2 \).

4. \( X_{t_1}^{s-ST} \geq X_{t_2}^{s-ST} \), for all \( 0 < t_1 \leq t_2 \).

5. \( X_{t_1}^{s-SFR} \geq X_{t_2}^{s-SFR} \), for all \( 0 < t_1 \leq t_2 \).

Proof. Let \( X \) have distribution \( F \) and \( X_t \) have distribution \( F_t \). With \( T_s \) and \( U_s \) we will mean the equilibrium distribution of order \( s \) of \( X \) and \( X_t \), respectively, and also let \( V_s \) and \( W_s \) we will mean the equilibrium distribution of order \( s \) of \( X_{t_1} \) and \( X_{t_2} \), respectively.

\( X_{t_1}^{s-ST} \geq X_{t_2}^{s-ST} \), \( 0 < t_1 \leq t_2 \) \( \iff \) \( \frac{T_s(t_1 + x)}{T_s(t_1)} \geq \frac{T_s(t_2 + x)}{T_s(t_2)} \), \( 0 < t_1 \leq t_2 \), \( x \geq 0 \)

\( \iff \) \( \frac{T_s(t_1 + x)}{T_s(t_2 + x)} \geq \frac{T_s(t_1)}{T_s(t_2)} \), \( 0 < t_1 \leq t_2 \), \( x \geq 0 \)

\( \iff \) \( \frac{T_s(x)}{T_s(t + x)} \) is nondecreasing in \( x \geq 0 \), for each \( t > 0 \)

\( \iff \) \( X^{s-FR} \geq X_t \), \( t > 0 \)

\( X^{s-FR} \geq X_t \), \( t > 0 \) \( \iff \) \( -\frac{d}{dx} \frac{T_s(x + t)}{T_s(x)} \leq -\frac{d}{dx} \frac{U_s(x)}{U_s(x)} \), \( x \geq 0 \)

\( \iff \) \( r_{t_1}(x) \leq r_{t_2}(x) \), \( x \geq 0 \)

\( \iff \) \( r_{t_1}(x) \leq r_{t_2}(t + x) \), each \( t \), \( x \geq 0 \)

\( \iff \) \( r_{t_1}(x + t_1) \leq r_{t_1}(x + t_2) \), \( 0 < t_1 \leq t_2 \), \( x \geq 0 \)

\( \iff \) \( X_{t_1}^{s-FR} \geq X_{t_2}^{s-FR} \), \( 0 < t_1 \leq t_2 \).

\( X_{t_1}^{s-FR} \geq X_{t_2}^{s-FR} \), \( 0 < t_1 \leq t_2 \) \( \iff \) \( r_{t_1}(x + t_1) \leq r_{t_1}(x + t_2) \), \( 0 < t_1 \leq t_2 \), \( x \geq 0 \)
\[ \Leftrightarrow r_{i_1}(t_1) \leq r_{i_2}(t_2), \ 0 < t_1 \leq t_2 \]
\[ \Leftrightarrow r_{v_i}(0) \leq r_{w_i}(0) \]
\[ \Leftrightarrow X_{s-ST} \preceq X_{s-ST}, \ 0 < t_1 \leq t_2. \]

In the following corollary, we summarize the results regarding YBO and NBU type comparisons in terms of the various orders, and survey the relevance of Theorem 2.3 and classical orderings (see Deshpande et al(1986) and Choi et al(1995)).

**Corollary 2.4:**

1) \( X \) is 0-IFR (ILR) if and only if \( 0-\text{FR} \preceq 0-\text{ST} \preceq X_{s-ST} (X_{s-ST} \geq X_{s-ST}), t > 0 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \).

2) \( X \) is 1-IFR (IFR) if and only if \( 1-\text{FR} \preceq 1-\text{ST} \preceq X_{s-ST} (X_{s-ST} \geq X_{s-ST}), t > 0 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \).

3) \( X \) is 2-IFR (DMRL) if and only if \( 2-\text{FR} \preceq 2-\text{ST} \preceq X_{s-ST} (X_{s-ST} \geq X_{s-ST}), t > 0 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \) if and only if \( X_{s-ST} \geq X_{s-ST} (X_{s-ST} \geq X_{s-ST}) \) holds for all \( 0 < t_1 \leq t_2 \).

Next, we consider that the YBO and NBU comparisons in terms of \( s-ST \) ordering and \( s-SFR \) ordering.

**Theorem 2.5:** If \( X_{s-ST} \geq X_{s-ST}, \ 0 < t_1 \leq t_2, \) then \( X_{s-ST} \geq X_{s-ST}, \ t > 0. \)

**Proof.** Using Theorem 2.3 and the fact that \( X \geq Y \) implies \( X \geq Y. \)
Some Partial Orders

Theorem 2.6: If $X^s_{t_1} \geq X^s_{t_2}$, $0 < t_1 \leq t_2$, then $X^s_{t_1} \geq X^s_{t_2}$, $t > 0$.

Proof. Using Theorem 2.3 and the fact that $X \geq Y$ implies $X^s_{t} \geq Y^s_{t}$.

In the following corollary, we survey connections of Theorem 2.5, Theorem 2.6 and classical orderings (see Deschpande et al (1986)).

Corollary 2.7:

1) If $X^1_{t_1} \geq X^1_{t_2}$ ($X^1_{t_1} \geq X^1_{t_2}$), $0 < t_1 \leq t_2$, then $X^1_{t_1} \geq X^1_{t_2}$ ($X^1_{t_1} \geq X^1_{t_2}$), $t > 0$.

2) If $X^1_{t_1} \geq X^1_{t_2}$ ($X^1_{t_1} \geq X^1_{t_2}$), $0 < t_1 \leq t_2$, then $X^1_{t_1} \geq X^1_{t_2} (X^1_{t_1} \geq X^1_{t_2})$, $0 < t_1 \leq t_2$.

3) If $X^1_{t_1} \geq X^1_{t_2}$ ($X^1_{t_1} \geq X^1_{t_2}$), $0 < t_1 \leq t_2$, then $X^2_{t_1} \geq X^2_{t_2} (X^2_{t_1} \geq X^2_{t_2})$, $t > 0$.

3. EQUILIBRIUM DISTRIBUTIONS

We consider a renewal process as generated by putting as item, with life distribution $F$, in use and on replacing it upon failure, the limiting distribution of residual life has distribution function $H_F(x) = \frac{1}{\mu_F} \int_0^x \overline{F}(u) du$, and density function $h_F(x) = \frac{\overline{F}(x)}{\mu_F}$. Let $Z$ denote the random variable density $h_F(x)$ and $Z_t = (Z - t) I(Z > t)$ for any $t \geq 0$.

Using the fact the equilibrium distributions of order $(s+1)$ of $X$ is the equilibrium distribution of $s$-order equilibrium distribution, we have the following theorem.

Theorem 3.1:

1) $X^{(s+1)}_{t} \geq X^s_{t}$, $t > 0$ if and only if $Z \geq Z^s_{t}$, $t > 0$. 

2) $X \geq X_t, \ t > 0$ if and only if $Z \geq Z_t, \ t > 0$

In Reference Singh(1989) and Choi et al(1995), various positive ageing properties of $X$ have been obtained in terms of positive ageing properties of $Z$. The following corollary gives the relevance of Theorem 3.1 and classical orderings.

**Corollary 3.2:** For all $t > 0$,

1) $X \geq X_t (X \geq X_t), \ t > 0$ if and only if $Z \geq Z_t (Z \geq Z_t), \ t > 0$.

2) $X \geq X_t (X \geq X_t), \ t > 0$ if and only if $Z \geq Z_t (Z \geq Z_t), \ t > 0$.

3) $X \geq X_t (X \geq X_t), \ t > 0$ if and only if $Z \geq Z_t (Z \geq Z_t), \ t > 0$.

4) $X \geq X_t (X \geq X_t), \ t > 0$ if and only if $Z \geq Z_t (Z \geq Z_t), \ t > 0$.

5) $X \geq X_t (X \geq X_t), \ t > 0$ if and only if $Z \geq Z_t (Z \geq Z_t), \ t > 0$.

**References**


Some Partial Orders

Inferences, ~16, 329 - 335.