STRONG LAW OF LARGE NUMBERS
FOR LEVEL-WISE INDEPENDENT
FUZZY RANDOM VARIABLES

YUN KYONG KIM

ABSTRACT. In this paper, we obtain a strong law of large numbers for sums of level-wise independent and level-wise identically distributed fuzzy random variables.

1. Introduction

Laws of large numbers for sums of independent random sets have been studied by Artstein and Hart [1], Artstein and Vitale [2], Puri and Ralescu [16], Taylor and Inoue [18], Uemura [19], etc. These results have been generalized to the case of fuzzy random variables by several people. A SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Kruse [14], and a SLLN for sums of independent fuzzy random variables by Miyakoshi and Shimbo [15]. Also, Klement, Puri and Ralescu [12] proved some limit theorems which includes a SLLN, and Inoue [10] obtained a SLLN for sums of independent tight fuzzy random sets. Recently, Hong and Kim [9] generalized Marcinkiewicz law of large numbers to fuzzy random variables.

In this paper, we obtain a SLLN for sums of level-wise independent and level-wise identically distributed fuzzy random variables by using a metric which is stronger than one in works mentioned previously. The representation theorem of fuzzy numbers by Goetschel and Voxman [7] will be used.

1991 Mathematics Subject Classification: 60D05, 60F15.
Key words and phrases: Strong law of large numbers, Fuzzy random variables, Level-wise independent.
This research was supported by the Dongshin University research grants in 1996.
2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers. Let $R$ denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \rightarrow [0, 1]$ with the following properties;

1. $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.
2. $\tilde{u}$ is upper semicontinuous.
3. $\text{supp} \; \tilde{u} = \text{cl} \{ x \in R : \tilde{u}(x) > 0 \}$ is compact.
4. $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0, 1]$.

For a fuzzy set $\tilde{u}$, we define

$$L_\alpha \tilde{u} = \left\{ \begin{array}{ll}
{x : \tilde{u}(x) \geq \alpha}, & 0 < \alpha \leq 1 \\
\text{supp} \; \tilde{u}, & \alpha = 0
\end{array} \right.$$

Then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0, 1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u^-_\alpha, u^+_\alpha]$. We denote the family of all fuzzy numbers by $F(R)$.

**Theorem 2.1 ([7]).** For $\tilde{u} \in F(R)$, we denote $u^-(\alpha) = u^-_\alpha$ and $u^+(\alpha) = u^+_\alpha$. Then the followings hold;

1. $u^-(\alpha)$ is a bounded increasing function on $[0,1]$.
2. $u^+(\alpha)$ is a bounded decreasing function on $[0,1]$.
3. $u^-(1) \leq u^+(1)$.
4. $u^-(\alpha)$ and $u^+(\alpha)$ are left continuous on $(0,1]$ and right continuous at 0.
5. If $v^-\alpha$ and $v^+\alpha$ satisfy above (1)-(4), then there exists unique $\tilde{v} \in F(R)$ such that $v^-\alpha = v^-\alpha$, $v^+\alpha = v^+\alpha$.

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u^-\alpha, u^+\alpha)| 0 \leq \alpha \leq 1\}$. Suppose now that $\tilde{u}, \tilde{v}$ are fuzzy numbers represented by $\{(u^-\alpha, u^+\alpha)| 0 \leq \alpha \leq 1\}$ and $\{(v^-\alpha, v^+\alpha)| 0 \leq \alpha \leq 1\}$, respectively. If we define

$$(\tilde{u} + \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$
\[ (\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0 \end{cases} \]

where \( \tilde{0} = I_{\{0\}} \) is the indicator function of \( \{0\} \), then

\[ \tilde{u} + \tilde{v} = \{(u^- + v^- + v^+), 0 \leq \alpha \leq 1\}, \]

\[ \lambda \tilde{u} = \begin{cases} \{(\lambda u^- + \lambda u^+), 0 \leq \alpha \leq 1\}, & \lambda \geq 0 \\ \{(\lambda u^+ + \lambda u^-), 0 \leq \alpha \leq 1\}, & \lambda < 0. \end{cases} \]

Now, we define two metrics \( d, d^* \) on \( F(R) \) by

\[ d(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v}) \]

\[ d^*(\tilde{u}, \tilde{v}) = \int_0^1 d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v})d\alpha \]

where \( d_H \) is the Hausdorff metric defined as

\[ d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u^- - v^-|, |u^+ - v^+|). \]

Also, the norm \( \|\tilde{u}\| \) of fuzzy number \( \tilde{u} \) will be defined as

\[ \|\tilde{u}\| = d(\tilde{u}, \tilde{0}) = \max(|u^-|, |u^+|). \]

### 3. Fuzzy random variables

Throughout this paper, \((\Omega, A, P)\) denotes a complete probability space. If \( \tilde{X} : \Omega \to F(R) \) is a fuzzy number valued function and \( B \) is a subset of \( R \), then \( \tilde{X}^{-1}(B) \) denotes the fuzzy subset of \( \Omega \) defined by

\[ \tilde{X}^{-1}(B)(\omega) = \sup_{x \in B} \tilde{X}(\omega)(x) \]

for every \( \omega \in \Omega \). The function \( \tilde{X} : \Omega \to F(R) \) is called a fuzzy random variable if for every closed subset \( B \) of \( R \), the fuzzy set \( \tilde{X}^{-1}(B) \) is measurable when considered as a function from \( \Omega \) to \([0, 1]\). If we denote \( \tilde{X}(\omega) = \{(X^-_\alpha(\omega), X^+_\alpha(\omega)) | 0 \leq \alpha \leq 1\} \), then it is well-known that \( \tilde{X} \) is a fuzzy random variable if and only if for each \( \alpha \in [0, 1] \), \( X^-_\alpha \) and \( X^+_\alpha \) are random variables in the usual sense (See Kim and Ghil [11]). Hence, if \( \sigma(\tilde{X}) \) is the smallest \( \sigma \)-field which makes \( \tilde{X} \) a fuzzy random variable, then \( \sigma(\tilde{X}) \) is consistent with \( \sigma(\{X^-_\alpha, X^+_\alpha | 0 \leq \alpha \leq 1\}) \). This enables us to define the concept of independence for fuzzy random variables as in the case of classical random variables.
Definition 3.1. Let $\tilde{X}, \tilde{Y}$ be two fuzzy random variables whose representations are $\{(X_\alpha^-, X_\alpha^+) | 0 \leq \alpha \leq 1\}$ and $\{(Y_\alpha^-, Y_\alpha^+) | 0 \leq \alpha \leq 1\}$, respectively.

1. $\tilde{X}$ and $\tilde{Y}$ are called independent if the $\sigma$-fields $\sigma(\tilde{X})$ and $\sigma(\tilde{Y})$ are independent.
2. $\tilde{X}$ and $\tilde{Y}$ are called level-wise independent if for each $\alpha \in [0, 1]$, the $\sigma$-fields $\sigma(X_\alpha^-, X_\alpha^+)$ and $\sigma(Y_\alpha^-, Y_\alpha^+)$ are independent.
3. $\tilde{X}$ and $\tilde{Y}$ are called level-wise identically distributed if for each $\alpha \in [0, 1]$, $(X_\alpha^-, X_\alpha^+)$ and $(Y_\alpha^-, Y_\alpha^+)$ are identically distributed random vectors.

Note that the definitions (2) and (3) is firstly introduced in this paper.

Definition 3.2. A fuzzy random variable $\tilde{X} = \{X_\alpha^- - X_\alpha^+ | 0 \leq \alpha \leq 1\}$ is called integrable if for each $\alpha \in [0, 1]$, $X_\alpha^-$ and $X_\alpha^+$ are integrable, equivalently, $\int \|\tilde{X}\|dP < \infty$. In this case, the expectation of $\tilde{X}$ is defined by

$$E\tilde{X} = \int \tilde{X}dP = \{(\int X_\alpha^- dP, \int X_\alpha^+ dP) | 0 \leq \alpha \leq 1\}$$

4. Main Result

In this section, a SLLN with respect to the metric $d$ defined as in (2.1) will be obtained. In earlier works, the metric $d^*$ defined as in (2.2) have been used (see [9],[10],[12]). Note that $d$ is stronger than $d^*$. First, we need a subspace $F_C(R)$ of $F(R)$. Let $F_C(R) = \{\tilde{u} \in F(R) | u^-_\alpha$ and $u^+_\alpha$ are continuous when considered as functions of $\alpha\}$. Then it is known that $\tilde{u} \in F_C(R)$ if and only if for any $\beta \in (0, 1)$, there exist at most two different $x_1, x_2$ such that $\tilde{u}(x_1) = \tilde{u}(x_2) = \beta$ (See [4] Theorem 5.1). Note that if $\tilde{X}$ is $F_C(R)$-valued, then $E\tilde{X} \in F_C(R)$.

Before we state the main result, we recall the following lemma which is well-known in the classical Analysis.

Lemma 4.1. Let $(f_n)$ be a sequence of monotonic functions on $[0, 1]$. If $f_n(x)$ converges pointwise to a continuous function $f(x)$ on $[0, 1]$, then $f_n(x)$ converges to $f(x)$ uniformly.

We now state the SLLN for sums of level-wise independent fuzzy random variables.
THEOREM 4.2. Let \(\tilde{X}_n\) be a sequence of level-wise independent and level-wise identically distributed fuzzy random variables with \(E\|\tilde{X}_1\| < \infty\). If \(E\tilde{X}_1 \in F_C(R)\), then

\[
d\left(\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1 \right) \rightarrow 0 \quad \text{a.s.}
\]

PROOF. Let \(\tilde{X}_n = \{(X^-_{n\alpha}, X^+_{n\alpha})| 0 \leq \alpha \leq 1\}\). Then for each \(\alpha \in [0, 1]\), \(\{(X^-_{n\alpha}, X^+_{n\alpha})\}\) is a sequence of independent and identically distributed random vectors with \(E|X^-_{n\alpha}| < \infty\) and \(E|X^+_{n\alpha}| < \infty\) in the classical sense. By Kolmogorov’s strong law of large numbers,

\[
\frac{1}{n} \sum_{i=1}^{n} X^-_{i\alpha} \rightarrow E X^-_{1\alpha} \quad \text{a.s.}
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} X^+_{i\alpha} \rightarrow E X^+_{1\alpha} \quad \text{a.s.}
\]

Now, let \(\{r_k\}\) be a countable dense subset of \([0, 1]\) with \(r_0 = 0, r_1 = 1\). Then there exist \(B_k \in A\) with \(P(B_k) = 0\) such that for each \(\omega \notin B_k\)

\[
\begin{align*}
(4.1) \quad & \frac{1}{n} \sum_{i=1}^{n} X^-_{i r_k}(\omega) \rightarrow E X^-_{1 r_k} \\
(4.2) \quad & \frac{1}{n} \sum_{i=1}^{n} X^+_{i r_k}(\omega) \rightarrow E X^+_{1 r_k}
\end{align*}
\]

If we define \(B = \bigcup_{k=0}^{\infty} B_k\), then \(P(B) = 0\) and for each \(\omega \notin B\), (4.1) and (4.2) hold for all \(r_k\). Now, we will show that for each \(\omega \notin B\)

\[
\frac{1}{n} \sum_{i=1}^{n} X^-_{i\alpha}(\omega) \rightarrow E X^-_{1\alpha} \quad \text{uniformly in } \alpha \in [0, 1].
\]
By Lemma 4.1, it suffices to show that for each \( \omega \not\in B \), and each \( \alpha \),

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) \longrightarrow EX_{1\alpha}^-.
\]

Let \( \omega \not\in B \) and \( \epsilon > 0 \) be fixed. Then by the continuity of \( EX_{1\alpha}^- \) as a function of \( \alpha \), there exists \( \delta > 0 \) such that

\[
|\alpha - \beta| < \delta \quad \text{implies} \quad |EX_{1\alpha}^- - EX_{1\beta}^-| < \epsilon
\]

If we take \( r_l, r_m \) so that \( \alpha - \delta < r_l < \alpha < r_m < \alpha + \delta \), then

\[
EX_{1r_m}^- - \epsilon < EX_{1\alpha}^- < EX_{1r_l}^- + \epsilon.
\]

Hence, by the monotonicity of \( X_{i\alpha}^- (\omega) \) with respect to \( \alpha \),

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) - EX_{1\alpha}^- - \epsilon < \frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) - EX_{1\alpha}^-
\]

\[
< \frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) - EX_{1r_m}^- + \epsilon
\]

which implies

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) \longrightarrow EX_{1\alpha}^-.
\]

Similarly, it can be proved that for each \( \omega \not\in B \)

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^+ (\omega) \longrightarrow EX_{1\alpha}^+ \quad \text{uniformly in} \quad \alpha \in [0, 1].
\]

Therefore, for each \( \omega \not\in B \).

\[
d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i (\omega), E\tilde{X}_1 \right) \longrightarrow 0.
\]
COROLLARY 4.3. Let \( \{\tilde{X}_n\} \) be a sequence of level-wise independent and level-wise identically distributed \( F_C(R) \)-valued fuzzy random variables. There exists \( \tilde{b} \in F_C(R) \) such that

\[
(4.3) \quad d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \tilde{b} \right) \longrightarrow 0 \quad a.s.
\]

if and only if \( E\|\tilde{X}_1\| < \infty \). Furthermore, if (4.3) holds, then \( \tilde{b} = E\tilde{X}_1 \).

PROOF. The sufficiency follows immediately from theorem 4.2. To prove the converse, if (4.3) holds, then for any \( \alpha \in [0,1] \),

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- \longrightarrow b^-_\alpha \quad a.s.
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} X^+_{i\alpha} \longrightarrow b^+_\alpha \quad a.s.
\]

By the converse of Kolmogorov's strong law of large numbers,

\[
E|X^-_{1\alpha}| < \infty, E|X^+_{1\alpha}| < \infty \quad \text{for each} \quad \alpha \in [0,1]
\]

which implies \( E\|\tilde{X}_1\| < \infty \) and \( \tilde{b} = E\tilde{X}_1 \).

EXAMPLE. Let \( \tilde{u} \in F_C(R) \) be fixed and let \( \{Y_n\} \) be i.i.d. with \( E|Y_1| < \infty \) in the usual sense. Define \( \tilde{X}_n(w)(x) = \tilde{u}(x - Y_n(w)) \) i.e., \( \tilde{X}_n(w) \) is the translation of \( \tilde{u} \) by \( Y_n(w) \) in \( x \)-axis. Then

\[
X^-_{n,\alpha}(w) = u^-_\alpha + Y_n(w) \quad \text{and} \quad X^+_{n,\alpha}(w) = u^+_\alpha + Y_n(w)
\]

Hence the above theorem implies that

\[
d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1 \right) \longrightarrow 0 \quad a.s.
\]

where \( (E\tilde{X}_1)(x) = \tilde{u}(x - EY_1) \).

As a final result, we give a generalization of Chung's SLLN to the case of fuzzy random variables.
THEOREM 4.4. Let \( \{\tilde{X}_n\} \) be a sequence of fuzzy random variables. If \( \{\|\tilde{X}_n\|\} \) are independent random variables in classical sense and

\[
\sum_{n=1}^{\infty} \frac{1}{n} E\|\tilde{X}_n\| < \infty,
\]

then

\[
d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n} \sum_{i=1}^{n} E\tilde{X}_i \right) \longrightarrow 0 \ a.s.
\]

PROOF. First we note that

\[
d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n} \sum_{i=1}^{n} E\tilde{X}_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} d(\tilde{X}_i, E\tilde{X}_i)
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\| + E\|\tilde{X}_i\|).
\]

Since \( \{\|\tilde{X}_n\|\} \) is a sequence of independent random variables, (4.4) and Chung's law of large numbers yields

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\| - E\|\tilde{X}_i\|) = 0 \ a.s.
\]

(4.5)

Now, applying the Kronecker lemma to (4.4), we obtain

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E\|\tilde{X}_i\| = 0
\]

which implies, together with (4.5),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\| = 0 \ a.s.
\]

This gives the desired result. \( \square \)
References


Department of Mathematics
Dongshin University
Chonnam 520-714, Korea