Initial L-fuzzy quasi-uniform structures

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Abstract

We prove the existence of initial $L$-fuzzy (quasi-)uniform structures. From this fact, we can define subspaces and products of $L$-fuzzy (quasi-)uniform spaces.

Key Words: $L$-fuzzy (quasi-)uniform structures, initial $L$-fuzzy (quasi-)uniform structures, subspaces and products of $L$-fuzzy (quasi-)uniform spaces.

1. Introduction


In this paper, we will prove the existence of initial $L$-fuzzy (quasi-)uniform structures. From this fact, we can define subspaces and products of $L$-fuzzy (quasi-)uniform spaces.

2. Preliminaries

In this paper, let $L$ be a nonempty set. Let $L = (L, \leq, \vee, \wedge, \cdot, ^{-1})$ be a completely distributive lattice with an order-reversing involution $^{-1}$ and 0 and 1 denote the least and the greatest element in $L$. For $\sigma \in L$, $\sigma(x) = \sigma$ for each $x \in L$ and $L_1 = L - \{1\}$.

Let $\Omega X$ denote the family of all functions $f: L^X \rightarrow L^X$ with the following properties:

1. $f(0) = 0$, $\mu \leq f(\mu)$, for every $\mu \in L^X$.
2. $f(\mu \wedge \nu) = f(\mu) \wedge f(\nu)$, for $\mu, \nu \in L^X$.

For $f, g \in \Omega X$, we define, for all $\mu \in L^X$,

$f^{-1}(\mu) = \{ \rho \in L^X \mid f(\rho) \leq \mu \}$,

$(f \circ g)(\mu) = g(f(\mu)), (f \cdot g)(\mu) = f(g(\mu))$.

Then $f^{-1}, f \circ g, f \cdot g \in \Omega X$.

Lemma 2.1 ([5-8]) For every $f, g, h, f_1, g_1 \in \Omega X$ the following properties hold:

1. If $f \leq f_1, g \leq g_1$, then $f \circ g \leq f_1 \circ g_1$.
2. $f \circ g \leq f$, $f \circ g \leq g$, $(f \circ g) \circ h = f \circ (g \circ h)$ and $f \circ f = f$.

Definition 2.2 ([7]) A function $U: \Omega X \rightarrow L$ is said to be an $L$-fuzzy quasi-uniformity on $X$ if it satisfies the following conditions:

QU1) $U(f_1 \circ f_2) \geq U(f_1) \wedge U(f_2)$, for all $f_1, f_2 \in \Omega X$.

QU2) For $f \in \Omega X$, we have $U(f_1) \vee U(f_2) \geq U(f_1 \cdot f_2)$. $U(f_1 \vee f_2) \geq U(f_1)$.

QU3) If $f \leq f_1$ then $U(f_1) \geq U(f)$.

QU4) There exists $f_0 \in \Omega X$ such that $U(f) = 1$.

The pair $(X, U)$ is said to be an $L$-fuzzy quasi-uniform space.

An $L$-fuzzy quasi-uniform space $(X, U)$ is called an $L$-fuzzy uniform space if it satisfies:

(U) For $f \in \Omega_X$, $U(f_1) \vee U(f_2) \geq U(f_1 \cdot f_2)$.

Let $U_1$ and $U_2$ be $L$-fuzzy (resp. quasi-uniformities) on $X$. We say $U_1$ is finer than $U_2$, or $U_2$ is coarser than $U_1$, denoted by $U_2 \leq U_1$, iff for any $f \in \Omega_X, U_2(f) \leq U_1(f)$.

Let $(X, U)$ be an $L$-fuzzy quasi-uniform space. We define for $f \in \Omega_X$, $U^{-1}(f) = (U(f))^{-1}$. From Lemma 2.1, we easily show that $U^{-1}$ is an $L$-fuzzy uniformity on $X$. 
Lemma 2.3 ([5-8]) Let \( \psi: X \rightarrow Y \) be a function. For each \( f \in \Omega^Y \), a function \( \psi^{-1}(f): L^X \rightarrow L^X \) is defined by, for all \( \mu \in L^X \),

\[
\psi^{-1}(f)(\mu) = (\psi^{-1} \circ f \circ \psi^{-1})(\mu) = \psi^{-1}(f(\psi^{-1}(\mu))).
\]

For \( f, f_1, f_2 \in \Omega^Y \), we have the following properties.

1. \( \psi^{-1}(f) \in \Omega^X \).
2. If \( f_1 \leq f_2 \), then \( \psi^{-1}(f_1) \leq \psi^{-1}(f_2) \).
3. \( \psi^{-1}(f_1) \circ \psi^{-1}(f_2) \leq \psi^{-1}(f_1 \circ f_2) \) with equality if \( \psi \) is onto.
4. \( (\psi^{-1}(f))^{-1} = \psi^{-1}(f^{-1}) \).
5. \( \psi^{-1}(f_1) \circ \psi^{-1}(f_2) = \psi^{-1}(f_1 \circ f_2) \).
6. \( \psi^{-1}((\psi^{-1}(f))^{-1}(A)) \leq \psi^{-1}(f^{-1}(A)) \), for all \( A \in L^X \).
7. \( \psi^{-1}(f) = f_\psi \circ (\psi^{-1}) \).

Definition 2.4 ([7]) Let \( (X, U) \) and \( (Y, V) \) be \( L \)-fuzzy (quasi)-uniform spaces. A function \( \psi: X \rightarrow (Y, V) \) is \( LF \)-uniformly continuous if \( V(f) \leq U(\psi^{-1}(f)) \), for every \( f \in \Omega^Y \).

Theorem 2.5 ([7]) Let \( (X, U) \) and \( (Y, V) \) and \( (Z, W) \) be \( L \)-fuzzy (quasi)-uniform spaces. If \( \psi: X \rightarrow (Y, V) \) and \( \phi: (Y, V) \rightarrow (Z, W) \) are \( LF \)-uniformly continuous, then \( \phi \circ \psi: X \rightarrow (Z, W) \) is \( LF \)-uniformly continuous.

Theorem 2.6 ([7]) Let \( (X, U) \) and \( (Y, V) \) be \( L \)-fuzzy (quasi)-uniform spaces. If \( \psi: (X, U) \rightarrow (Y, V) \) is \( LF \)-uniformly continuous, then \( \psi(X, U^{-1}) \rightarrow (Y, V^{-1}) \) is \( LF \)-uniformly continuous.

3. Initial \( L \)-fuzzy quasi-uniformity structures

Theorem 3.1 Let \( \{(X_k, V_k) \mid k \in K\} \) be a family of \( L \)-fuzzy (resp. quasi-uniform) spaces, \( X \) a set and for each \( k \in K \), \( \phi_k: X \rightarrow X_k \) a function. We define a function \( U \circ \Omega_X \rightarrow L \)

\[
U(f) = \bigvee \left( \bigwedge_{k \in K} V_k(f_k) \cup \bigvee_{k \in K} \psi_k^{-1}(f_k) \right)
\]

where the \( \bigvee \) is taken over every finite index \( K = \{k_1, \ldots, k_n\} \subseteq K \).

(1) The structure \( U \) is the coarsest \( L \)-fuzzy (resp. quasi-uniformity) on \( X \) for which each \( \phi_k \) is \( LF \)-uniformly continuous.

(2) A map \( f: (Z, W) \rightarrow (X, U) \) is \( LF \)-uniformly continuous iff for each \( k \in K \), \( \phi_k \circ f: (Z, W) \rightarrow (X_k, V_k) \) is \( LF \)-uniformly continuous.

Proof (1) First, we will show that \( U \) is an \( L \)-fuzzy (resp. quasi-uniformity) on \( X \).

(QU1) Suppose there exists \( f, g \in \Omega_X \) such that

\[
U(f \circ g) \geq U(f) \wedge U(g).
\]

Since \( L \) is a completely distributive lattice, by the definition of \( U(f) \), there exists a finite index set \( K = \{k_1, \ldots, k_n\} \subseteq K \) such that

\[
U(f \circ g) \geq \bigvee_{k \in K} V_k(f_k) \wedge U(g), \quad \bigvee_{k \in K} \psi_k^{-1}(f_k) \leq f.
\]

Also, by definition of \( U(g) \), there exists a finite index set \( L = \{l_1, \ldots, l_m\} \subseteq L \) such that

\[
U(f \circ g) \geq \bigvee_{l \in L} V_l(f_l) \wedge U(g), \quad \bigvee_{l \in L} \psi_l^{-1}(f_l) \leq g.
\]

Since \( \bigvee_{l \in L} \psi_l^{-1}(f_l) \leq \bigvee_{k \in K} \psi_k^{-1}(f_k) \), we have

\[
U(f \circ g) \geq \bigvee_{l \in L} V_l(f_l) \wedge \bigvee_{k \in K} V_k(f_k), \quad \bigvee_{l \in L} \psi_l^{-1}(f_l) \leq g.
\]

It is a contradiction. Hence \( U(f \circ g) \geq U(f) \wedge U(g) \) for all \( f, g \in \Omega_X \).

(QU2) Suppose there exists \( f \in \Omega_X \) such that

\[
U(f) \mid f_1 \leq f \leq f_2 \leq U(f).
\]

By the definition of \( U(f) \), there exists a finite index \( K = \{k_1, \ldots, k_n\} \subseteq K \) such that

\[
U(f) \mid f_1 \leq f \leq f_2 \leq U(f).
\]

For each \( k \in K \), since \( (X_k, V_k) \) is an \( L \)-fuzzy (resp. quasi-uniform) space, by (QU2),

\[
\bigvee_{k \in K} V_k(f_k) \mid f_1 \leq f \leq f_2 \leq \bigvee_{k \in K} V_k(f_k).
\]

Since \( L \) is a completely distributive lattice, for each \( k \in K \), there exists \( g_k \in \Omega_X \) with \( g_k \leq f_k \), such that

\[
\bigvee_{k \in K} V_k(f_k) \mid g_k \leq f_k \leq g_k \leq \bigvee_{k \in K} V_k(f_k).
\]

Put \( g = \bigvee_{k \in K} \psi_k^{-1}(g_k) \). For each \( k \in K \), we have

\[
g \leq \psi_k^{-1}(g_k) \leq \psi_k^{-1}(g_k).
\]

Hence, by Lemma 2.1 (1) and (2),

\[
g \leq \psi_k^{-1}(g_k) \leq \psi_k^{-1}(g_k) \leq f_k ( \text{by Lemma 2.3(3)}).
\]

Then we have \( g \leq f \) and

\[
U(g) \geq \bigvee_{k \in K} V_k(g_k).
\]

It is a contradiction. Hence \( U(f) \mid f_1 \leq f \leq f_2 \leq U(f) \), for all \( f \in \Omega_X \).

(QU3) It is trivial.
(QU4) Since each \( (X, V, \psi) \) is an \( L \)-fuzzy (resp. quasi-)uniform space, by (QU4), there exists \( f_i \in \mathcal{O}_X \), such that \( V_k(f_i) > 0 \). For all finite indices \( K = \{k_1, \ldots, k_n\} \subseteq \Gamma \), put \( f = \vee_{i=1}^n \psi_k(f_{k_i}) \). Then there exists \( f \in \mathcal{O}_X \) such that \( U(f) = 1 \).

(U) Let \( \{ (X_k, U_k) | k \in \Gamma \} \) be a family of \( L \)-fuzzy uniform spaces. Suppose that there exists \( f \in \mathcal{O}_X \) such that
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
By the definition of \( U \), there exists a finite index \( K = \{k_1, \ldots, k_n\} \subseteq \Gamma \) such that
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
For each \( k \in K \), since \( (X_k, V_k) \) is an \( L \)-fuzzy uniform space, by (U),
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
For each \( k \in K \), there exists \( g_k \in \mathcal{O}_{X_k} \), with \( g_k \leq f_{k_i} \), such that
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
On the other hand, we have
\[
\bigwedge_{i=1}^n \psi_k(g_k) \leq \bigwedge_{i=1}^n \psi_k(f_{k_i}) \quad \text{(by Lemma 2.2(2))}
\]
\[
= \bigwedge_{i=1}^n (\psi_k(f_{k_i}))^{-1} \quad \text{(by Lemma 2.3(4))}
\]
\[
\leq \bigwedge_{i=1}^n (\psi_k(f_{k_i})) \quad \text{(by Lemma 2.1(6))}
\]
Put \( g = \bigwedge_{i=1}^n \psi_k(g_k) \). Then there exists \( g \in \mathcal{O}_X \) such that
\[
g \leq f, \quad U(g) \supseteq \bigwedge_{i=1}^n V_k(g_k).
\]
Thus
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(g).
\]
It is a contradiction. Hence \( \bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f) \), for all \( f \in \mathcal{O}_X \).

Second, by the definition of \( U \), for all \( k \in \Gamma \), \( f_k \in \mathcal{O}_X \),
\[
U(f_k) \geq V_k(f_k).
\]
Hence each \( \phi : (X, U) \rightarrow (X, V, \psi) \) is \( LF \)-uniformly continuous.

Finally, if \( \phi : (X, U) \rightarrow (X, V, \psi) \) is \( LF \)-uniformly continuous, that is, \( U^* (\psi_k(f_{k_i})) \geq V_k(f_k) \) for all \( k \in \Gamma \), then it is proved that \( U^* \geq U \) from the following:
for all \( f \in \mathcal{O}_X \)
\[
U(f) = \bigwedge_{i=1}^n V_k(f_{k_i}) \quad \text{and} \quad \bigwedge_{i=1}^n \psi_k(f_{k_i}) \leq f
\]
\[
\leq \bigwedge_{i=1}^n U^* (\psi_k(f_{k_i})) \quad \text{or} \quad \bigwedge_{i=1}^n \psi_k(f_{k_i}) \leq f
\]
\[
\leq \bigwedge_{i=1}^n U^* (\bigwedge_{i=1}^n \psi_k(f_{k_i})) \quad \text{or} \quad \bigwedge_{i=1}^n \psi_k(f_{k_i}) \leq f
\]
\[
\leq U^* (f).
\]

(2) Necessity of the composition condition is clear since the composition of \( LF \)-uniformly continuous maps is \( LF \)-uniformly continuous.

Conversely, suppose that \( \phi : (X, U) \rightarrow (X, V, \psi) \) is \( LF \)-uniformly continuous. There exists \( f \in \mathcal{O}_X \) such that
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
By the definition of \( U \), there exists a finite index set \( K = \{k_1, \ldots, k_n\} \subseteq \Gamma \) such that
\[
\bigwedge_{i=1}^n V_k(f_{k_i}) \subseteq U(f).
\]
On the other hand, for each \( k \in K \), since \( \psi_k \circ \phi \) is \( LF \)-uniformly continuous, we have
\[
V_k(f_k) \leq \bigwedge_{i=1}^n V_k(f_{k_i}) \quad \text{(by Lemma 2.3(4))}
\]
\[
\leq \bigwedge_{i=1}^n (\psi_k(f_{k_i})) \quad \text{(by Lemma 2.1(6))}
\]
\[
\leq \bigwedge_{i=1}^n (\psi_k(f_{k_i})) \quad \text{(by Lemma 2.1(4))}
\]
\[
\leq \bigwedge_{i=1}^n \psi_k(f_{k_i}) \quad \text{(by Lemma 2.3(3))}
\]
\[
\leq 0 \quad \text{(by Lemma 2.3(3))}
\]
It is a contradiction.

The category of \( LF \)-fuzzy uniform spaces and \( LF \)-uniformly continuous maps is denoted by \( L \text{-UNIF} \).

**Theorem 3.2** The forgetful functor \( W \colon L \text{-UNIF} \rightarrow \text{Set} \) defined by \( W(X, U) = X \) and \( W(\phi) = \phi \) is topological.

**Proof** From Theorem 3.1, every \( W \)-structured space \( (\phi : (X, U) \rightarrow (X, V, \psi)) \) has an unique \( W \)-initial lift \( (\phi_i : (X, U_i) \rightarrow (X, V, \psi)) \).

Using Theorems 3.1 and 3.2 and Definition 2.5, we obtain the following definition.

**Definition 3.3** Let \( \{ (X_i, U_i) | i \in \Gamma \} \) be a family of \( L \)-fuzzy (resp. quasi-)uniform spaces, \( X \) a set and \( \phi : X \rightarrow X \), a function, for each \( i \in \Gamma \): The initial \( L \)-fuzzy (resp. quasi-) uniform structure \( U \) on \( X \) with respect to \( (X, \psi_i, (X_i, U_i), \Gamma) \) is the coarsest \( L \)-fuzzy (resp. quasi-) uniform structure on \( X \) for which all \( \psi_i, i \in \Gamma \), are \( L \)-uniformly continuous maps.

**Definition 3.4** Let \( (X, V, \psi) \) be an \( L \)-fuzzy (resp. quasi-)uniform space and \( A \) a subset of \( X \). The pair \( (A, V, \psi) \) is said to be a subspace of \( (X, V, \psi) \) if it is endowed with the initial \( L \)-fuzzy (resp. quasi-)uniformity structure with respect to \( (A, i, (X, V)) \) where \( i \) is the inclusion map.
Definition 3.5 Let \( X = \prod_{i \in I} X_i \) be the product of sets from the family \( \{(X_i, U_i) \mid i \in I\} \) of \( L \)-fuzzy (resp. quasi-uniform) spaces. The initial \( L \)-fuzzy (resp. quasi-uniformity) structure \( U = \bigotimes U_i \) on \( X \) with respect to the family \( \{\pi_i : X \to (X_i, U_i) \mid i \in I\} \) of all projection maps is called the product \( L \)-fuzzy (resp. quasi-)uniformity structure of \( \{U_i \mid i \in I\} \), and \( (X, \bigotimes U) \) is called the product \( L \)-fuzzy (resp. quasi-)uniform space.

Corollary 3.6 Let \( (X, U) \) be a family of \( L \)-fuzzy (resp. quasi-uniform) spaces. We define, for \( f \in \mathcal{F}_X \),

\[
U(f) = \bigvee \left( \prod_{k \in K} U_k(f_k) \mid \forall i, f_k \leq f \right),
\]

where the \( \bigvee \) is taken over every finite index \( K = (k_1, \ldots, k_n) \subseteq I \). Then the structure \( U \) is the coarsest \( L \)-fuzzy (resp. quasi-)uniformity on \( X \) finer than \( U_f \).

References