Some generalized weak vector quasivariational-like inequalities for fuzzy mappings

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Abstract

Some Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings was introduced and the existence of solutions to them under non-compact assumption was considered using the particular form of the generalized Ky Fan’s section theorem due to Park [15]. As a corollary, Stampacchia type of generalized vector quasivariational-like inequalities for fuzzy mappings was studied under compact assumption using Ky Fan’s section theorem [7].

Key words : Fuzzy mapping, weak vector quasivariational-like inequality, generalized Ky Fan’s section theorem, topologically open fuzzy set-valued, weakly open fuzzy set-valued

1. Introduction

Since Chang and Zhu [5] firstly introduced a variational inequality problem for fuzzy mappings, many authors [2, 3, 8, 10-13] considered vector variational inequality problems for fuzzy mappings. In particular, Chang et al. studied vector quasivariational inequalities [2, 3] and vector variational-like inequalities [1] for fuzzy mappings and Lee et al. [10] obtained a fuzzy extension of Siddiqi et al.’s results [16] for vector variational-like inequalities. In [8], the authors considered the existence of solutions to generalized fuzzy vector quasivariational-like inequalities (F-VQVI) under the compact assumption. And then, in [9], the authors considered the weak case of the results in [8] under the non-compact assumption.

In this paper, we recall a Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings (F-WVQVI) and re-consider the existence of solutions to them under the non-compact assumption and to (F-VQVI) under the compact assumption considered in [8, 9].

Throughout this paper, $X$ denotes a Hausdorff topological vector space, $Y$ is a topological vector space, $Z$ is an ordered topological vector space, $\mathcal{S}(Y)$ is the collection of all fuzzy sets in $Y$ and $L(X, Z)$ is the set of all linear continuous mappings from $X$ to $Z$. Let $K$ be a nonempty convex subset of $X$, $D$ be a nonempty subset of $Y$ and $\{C(x) : x \in K\}$ be a family of solid convex cones in $Z$, that is, for any $x \in K$, $\text{int } C(x) \neq \emptyset$ and $C(x) \neq Z$. Let $F : K \to \mathcal{S}(D)$ and $G : K \to \mathcal{S}(K)$ be fuzzy mappings, $M : K \times D \to 2^{L(X, Z)}$ and $H : K \times K \to 2^Z$ be multivalued mappings, $\eta : X \times X \to X$ be a mapping, $\beta : X \to (0, 1]$ be a function and $\gamma$ be a constant in $(0, 1]$. $(F_x)_{\beta(x)} := \{d \in D : F_x(d) \geq \beta(x)\}$ is a cut set and $[F_x]_{\gamma(x)} := \{d \in D : F_x(d) > \beta(x)\}$ is a strong cut set.

An ordering $\leq$ with respect to the cone $C$ in $Z$ is defined as $y \not\leq_{\text{int } C} x$ if and only if $x - y \not\in -\text{int } C$ for $x, y \in Z$.

We consider the existence of solutions to the following Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings:

(F-WVQVI) Find $\bar{x} \in K$ such that there exists $\bar{s} \in (F_x)_{\beta(x)}$ satisfying the following inequality:

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \not\in -\text{int } C(\bar{x})$$

for any $x \in K$, $z \in (G_x)_x$, and $u \in H(x, \bar{x})$, where

$$\max \langle M(x, \bar{s}), \eta(x, z) \rangle = \max_{s \in M(\bar{x}, \bar{s})} \langle s, \eta(x, z) \rangle$$

and $\langle s, \eta(x, z) \rangle$ denotes the evaluation of continuous linear operator $s$ from $X$ into $Z$ at $\eta(x, z)$.

In addition, we obtain the existence of solutions to the following Stampacchia type of generalized vector quasivariational-like inequalities for fuzzy mappings:

(F-VQVI) Find $\bar{x} \in K$ such that, for any $x \in K$, there exists $\bar{s} \in (F_x)_{\beta(x)}$ satisfying the following inequality:

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \not\in -\text{int } C(\bar{x})$$

for any $z \in (G_x), g$, and $u \in H(x, \bar{x})$. 

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Replacing $\mathcal{O}(K)$ and $\mathcal{D}(D)$ with $2^K$ and $2^D$, respectively, in (F-WVQVL) and (F-VQVL), we obtain the following Stampacchia type of generalized weak vector variational-like inequalities and vector quasivariational-like inequalities for multivalued mappings:

**WVQVL** Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\{M(\bar{x}, \bar{s}), \eta(x, z)\} + u \notin -\text{int} C(\bar{x})$$

for any $x \in K$, $z \in G(x)$ and $u \in H(x, \bar{x})$.

**VQVL** Find $\bar{x} \in K$ such that, for any $x \in K$, that exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality:

$$\max\{M(\bar{x}, \bar{s}), \eta(x, z)\} + u \notin -\text{int} C(\bar{x})$$

for any $z \in G(x)$ and $u \in H(x, \bar{x})$.

Deleting a topological vector space $Y$ and a fuzzy mapping $F$, and then replacing $Z$ with an ordered topological vector space $Y$, $H : K \times K \rightarrow 2^Z$ with $H : K \times K \rightarrow Y$, and $M : K \times D \rightarrow 2^L(X, z)$ with $S : K \rightarrow 2^L(X, z)$ in (F-WVQVL) or (F-VQVL), respectively, we obtain the following vector variational-like inequality for fuzzy mappings:

**F-VVL** Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\{S(\bar{x}), \eta(x, y)\} + H(x, \bar{x}) \notin -\text{int} C(\bar{x})$$

for any $x \in K$ and any $y \in (G_x)(\beta(\bar{x}))$, where $\{C(x) : x \in K\}$ is a family of closed convex cones in $Y$.

Replacing a fuzzy mapping $G : K \rightarrow \mathcal{O}(K)$ with a multivalued mapping $G : K \rightarrow 2^K$ defined by $G(x) = K$ for $x \in K$ and putting $H \equiv 0$ in (F-VVL), we obtain the following vector variational-like inequalities for multivalued mappings:

**VVL** Find $\bar{x} \in K$ such that

$$\max\{S(\bar{x}), \eta(x, y)\} \notin -\text{int} C(\bar{x}), \quad x, y \in K,$$

which is a generalized form of the following vector variational-like inequalities for multivalued mappings introduced and studied by Chang, Thompson and Yuan [4]:

Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\{S(\bar{x}), \eta(x, \bar{x})\} \notin -\text{int} C(\bar{x}), \quad x \in K.$$

2. Preliminaries

**Definition 2.1.** ([6]) Let $X$, $Y$ be topological spaces and $T : X \rightarrow 2^Y$ be a multivalued mapping. Let $T^- : Y \rightarrow 2^X$ be a multivalued mapping defined by

$$x \in T^-(y) \quad \text{if and only if} \quad y \in T(x).$$

(1) $T$ is said to be upper semicontinuous (in short, u.s.c.) at $x \in X$ if, for every open set $V$ in $Y$ containing $T(x)$, there is an open set $U$ containing $x$ such that $T(u) \subseteq V$ for all $u \in U$;

(2) $T$ is said to be u.s.c. on $X$ if $T$ is u.s.c. at every point of $X$.

(3) $T$ is said to be lower semicontinuous (in short, l.s.c.) at $x \in X$ if, for every open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there is an open set $U$ containing $x$ such that $T(u) \cap V \neq \emptyset$ for all $u \in U$;

(4) $T$ is said to be l.s.c. on $X$ if $T$ is l.s.c. at every point of $X$.

(5) $T$ is said to be continuous at $x$ if $T$ is both u.s.c. and l.s.c. at $x$.

(6) $T$ is said to be compact if $T(X)$ is contained in some compact subset of $Y$.

(7) $T$ is said to be closed if the graph of $T$, $G_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

(8) $T$ is said to have open lower sections if for any $y \in Y$, $T^-(y)$ is open in $X$.

**Lemma 2.1.** $T$ is l.s.c. at $x \in X$ if, and only if, for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in $X$ converging to $x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for each $\alpha$, and $y_\alpha$ converging to $y$.

**Definition 2.2.** ([1, 2, 3, 5]) Let $X$, $Y$ be sets and $F : X \rightarrow \mathcal{O}(Y)$ be a fuzzy mapping. We denote a fuzzy set $F(x)$ by $F_x$ in $Y$ for $x \in X$.

(1) $F$ is said to be convex on a set $X$ if $Y$ is a convex subset of a topological vector space and for any $x \in X$, $y, z \in Y$ and $\lambda \in [0, 1]$,

$$F_x(\lambda y + (1 - \lambda)z) \geq \min\{F_x(y), F_x(z)\}.$$  

(2) $F$ is said to be closed fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.

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(3) \( F \) is said to be topologically open fuzzy set-valued if, for each \( x_0 \in X \) and for each open subset \( V \) of \( Y \) with \( F_{x_0}(y) \supseteq \gamma \) for some \( y \in V \) (\( \gamma \in (0, 1] \)), there is a neighborhood \( U \) of \( x_0 \) in \( X \) such that if \( x \in U \), then \( F_x(y) \supseteq \gamma \) for some \( y \in V \).

(4) \( F \) is said to be weakly open fuzzy set-valued if for each \( y \in Y, F_x(y) \) is l.s.c. on \( X \times Y \) as a real ordinary function.

**Lemma 2.2.** ([1]) Let \( K \) be a nonempty closed convex subset of a real Hausdorff topological vector space \( X, D \) a nonempty closed convex subset of a real Hausdorff topological vector space \( Y \), and \( \beta : K \to (0, 1] \) a l.s.c. function. Let \( F : K \to \mathbb{R}(D) \) be a fuzzy mapping with nonempty cut set \( (F_x)_{\beta(x)} \) for any \( x \in K \). Let \( \bar{F} : K \to \mathbb{R}^D \) be a multivalued mapping defined by \( \bar{F}(x) = (F_x)_{\beta(x)} \). If \( F \) is a convex fuzzy mapping with closed fuzzy set-values, then \( \bar{F} \) is a closed mapping with nonempty convex-values.

**Lemma 2.3.** ([2]) Let \( X \) and \( Y \) be topological spaces, and \( F : X \to \mathbb{R}(Y) \) be a fuzzy mapping such that for any \( x \in X \), the cut set \( (F_x)_\gamma \) is nonempty for \( \gamma \in (0, 1] \). Let \( \bar{F} : X \to 2^Y \) be a multivalued mapping defined by \( \bar{F}(x) = (F_x)_\gamma \). If \( F \) is convex and topologically open fuzzy set-valued, then \( \bar{F} \) is a l.s.c. mapping with nonempty convex-values.

**Lemma 2.4.** ([3]) Let \( K \) be a nonempty closed convex subset of a real Hausdorff topological vector space \( X, D \) a nonempty closed convex subset of a Hausdorff topological vector space \( Y \), and \( \beta : X \to (0, 1] \) an u.s.c. function. Let \( F : K \to \mathbb{R}(D) \) be a fuzzy mapping such that for any \( x \in K \), the strong cut set \( [F_x]_{\beta(x)} \) is nonempty. Let \( \bar{F} : K \to 2^D \) be a multivalued mapping defined by \( \bar{F}(x) = [F_x]_{\beta(x)} \).

1. If \( F \) is convex, then \( \bar{F} \) has nonempty convex-values.

2. If \( F \) is weakly open fuzzy set-valued, then \( \bar{F} \) has open lower sections.

**Definition 2.4.** Let \( X, Z \) be vector spaces. A mapping \( \eta : X \times X \to Z \) is said to be linear if

\[
\eta(\lambda(x_1, y_1) + (x_2, y_2)) = \lambda \eta(x_1, y_1) + \eta(x_2, y_2)
\]

for all \((x_1, x_2), (y_1, y_2) \in X \times X \) and \( \lambda \in R \).

**Definition 2.5.** ([18]) A point \( z_0 \in \) a nonempty subset \( C \) of \( Z \) is called a vector maximal point of \( C \) if the set \( \{ z \in C : z_0 \leq z, z \neq z_0 \} = \emptyset \), which is equivalent to

\[
C \cap (z_0 + P) = \{z_0\}.
\]

**Lemma 2.5.** ([14]) Let \( C \) be a nonempty compact subset of an ordered Banach space \( Z \). Then \( \max C \neq \emptyset \), where \( \max C \) denotes the set of all vector maximal points of \( C \).

The following particular form of the generalized Ky Fan's section theorem which is due to Park [15] will be used in dealing with (F-WVQVLI) for the noncompact set case.

**Theorem 2.1.** ([15]) Let \( K \) be a nonempty convex subset of \( X \) and \( A \subset K \times K \) satisfy the following conditions:

1. \((x, x) \in A, x \in K;\)
2. \((y \in K : (x, y) \in A), x \in K, \) is closed;
3. \((y \in K : (x, y) \notin A), y \in K, \) is convex or empty;
4. there exists a nonempty compact subset \( B \) of \( K \) such that for each finite subset \( N \) of \( K \) there exists a nonempty compact convex subset \( L_N \subset K \) containing \( N \) such that

\[
L_N \cap \{ y \in K : (x, y) \in A \text{ for any } x \in L_N \} \subset B. \]

Then there exists a \( y_0 \in B \) such that \( K \times \{y_0\} \subset A \).

In particular, if \( K = B \), that is, \( K \) is a compact convex subset of \( X \), then the condition (iv) is obviously true and thus we obtain Ky Fan's section theorem [7], which will be used in considering (F-VQVLI) under compact assumptions.
3. Main results

Now, we consider the existence of solutions to the Stampacchia type of (F-WVQVL) for non-compact set case studied in [9]. The following Proposition 3.1 is an existence theorem to a kind of vector variational-like inequalities for multi-valued mappings. With Lemma 3.1, it is an important tool for the following main results Theorem 3.1 and Theorem 3.2.

**Lemma 3.1.** (17) Let X be a paracompact Hausdorff topological space and Y be a topological vector space. Let \( F : X \to 2^Y \) be a multivalued mapping with nonempty convex-values. If \( F \) has open lower sections, then there exists a continuous function \( f : X \to Y \) such that \( f(x) \in F(x) \) for any \( x \in X \).

**Proposition 3.1.** Let \( K \) be a nonempty convex subset of \( X \) and \( D \) be a nonempty subset of \( Y \). Let \( f : K \to D \) be a continuous function and \( G : K \to 2^K \) be a l.s.c. multivalued mapping with convex-values. Let \( M : K \times D \to 2^{K(X, Z)} \) be a multivalued mapping, and a multivalued mapping \( W : K \to 2^Z \) defined by \( W(x) = Z\backslash \{ -\text{int} \ C(x) \} \), \( x \in K \), be closed. Let \( \eta : X \times X \to X \) be linear, \( \eta \) is continuous and \( H : K \times K \to 2^Z \) be \( P \)-convex with respect to the first variable and l.s.c. with respect to the second, where \( P = \bigcap_{x \in K} C(x) \). Suppose further that

(i) \( \max \{ M(y_n, s_n), \eta(x, z_n) \} \) converges to \( \max \{ M(y, s), \eta(x, z) \} \) provided that \( y_n \to y, s_n \to s \) and \( z_n \to z \);

(ii) \( \{ M(x, \cdot), \eta(x, \cdot) \} = 0 \) and \( H(x, x) = \emptyset \) for all \( x \in K \);

(iii) there is a nonempty compact subset \( B \) of \( K \) such that for each nonempty finite subset \( N \) of \( K \), there is a nonempty compact convex subset \( L_N \) of \( K \) containing \( N \) such that, for \( y \in L_N \backslash B \), there exist \( x \in L_N \), \( z \in G(y) \) and \( u \in H(x, z) \) such that

\[
\max \{ M(y, f(y)), \eta(x, z) \} + u \in -\text{int} \ C(y).
\]

Then there exists \( \bar{x} \in K \) such that

\[
\max \{ M(\bar{x}, f(\bar{x})), \eta(x, z) \} + u \notin -\text{int} \ C(\bar{x})
\]

for any \( x \in K, z \in G(\bar{x}) \) and \( u \in H(x, \bar{x}) \).

**Proof.** Let \( A = \{ (x, y) \in K \times K : \max \{ M(y, f(y)), \eta(x, z) \} + u \notin -\text{int} \ C(y) \} \) for any \( z \in G(y) \) and \( u \in H(x, y) \). It is easily shown that \( (x, z) \in A \) for \( x \in K \) from the condition (i). And \( A_x = \{ y \in K : (x, y) \in A \}, x \in K \), is closed. In fact, for any net \( \{ y_\alpha \} \in A_x \) converging to \( y \), we have \( \max \{ M(y_\alpha, f(y_\alpha)), \eta(x, z_\alpha) \} + u_\alpha \notin -\text{int} \ C(y_\alpha) \) for any \( z_\alpha \in G(y_\alpha) \) and \( u_\alpha \in H(x, y_\alpha) \). From Lemma 2.1 and the condition (i), \( \max \{ M(y, f(y)), \eta(x, z) \} + u \notin -\text{int} \ C(y) \) for any \( z \in G(y) \) and \( u \in H(x, y) \), so that we have \( y \in A_x \), which shows that \( A_x \) is closed for \( x \in K \). And \( A^p = \{ x \in K : (x, y) \notin A \}, y \in K \), is convex. Indeed, let \( x_1, x_2 \in A^p \) and \( \lambda \in [0, 1] \). Then, from the fact that \( (x_1, y) \notin A \) for any \( s \in F(y) \), there exist \( z_1 \in G(y) \) and \( u_1 \in H(x_1, y) \) such that

\[
\max \{ M(y, s), \eta(x_1, z_1) \} + u_1 \in -\text{int} \ C(y)
\]

and from the fact that \( (x_2, y) \notin A \) for any \( s \in F(y) \), there exist \( z_2 \in G(y) \) and \( u_2 \in H(x_2, y) \) such that

\[
\max \{ M(y, s), \eta(x_2, z_2) \} + u_2 \in -\text{int} \ C(y).
\]

On the other hand, from the convexity of \( G \), \( G \) is convex-valued due to Lemma 2.4(1). Hence, for any \( s \in F(y) \), there exist \( u \in H(x_1, (1 - \lambda)x_2, y) \) and \( z := \lambda z_1 + (1 - \lambda)z_2 \in G(y) \) for \( \lambda \in [0, 1] \) such that

\[
\max \{ M(y, s), \eta(x_1 + (1 - \lambda)x_2, z) \} + u
\]

\[
= \max \{ M(y, s), \eta(x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2) \}
\]

\[
+ u
\]

\[
= \max \{ M(y, s), \eta(x_1, z_1) \} + u
\]

\[
\leq \lambda \max \{ M(y, s), \eta(x_1, z_1) \} + (1 - \lambda) \max \{ M(y, s), \eta(x_2, z_2) \}
\]

\[
+ u
\]

\[
\in \lambda \max \{ M(y, s), \eta(x_1, z_1) \} + (1 - \lambda) \max \{ M(y, s), \eta(x_2, z_2) \}
\]

\[
+ \lambda u_1 + (1 - \lambda) u_2 - P
\]

\[
= \lambda (\max \{ M(y, s), \eta(x_1, z_1) \} + u_1)
\]

\[
+ (1 - \lambda) (\max \{ M(y, s), \eta(x_2, z_2) \} + u_2) - P
\]

\[
\subseteq -\text{int} \ C(y) - \text{int} C(y) - C(y)
\]

\[
= -\text{int} C(y).
\]

Thus \( \lambda z_1 + (1 - \lambda) z_2 \in A^p \), which shows that \( A^p \) is convex. Further, note that the condition (ii) implies that, for \( y \in L_N \backslash B \), there exists \( x \in L_N \) such that \( y \notin A_x \). Hence the condition (iv) of Theorem 2.1 is satisfied. Thus there exists \( \bar{x} \in K \) such that

\[
\max \{ M(\bar{x}, f(\bar{x})), \eta(x, z) \} + u \notin -\text{int} \ C(\bar{x})
\]

for any \( x \in K, z \in G(\bar{x}) \) and \( u \in H(x, \bar{x}) \). This completes the proof.

Now, we show the existence of solution for the problem (F-WVQVL1).
Theorem 3.1. Let $K$ be a nonempty paracompact convex subset of $X$ and $D$ be a nonempty closed convex subset of $Y$. Let $F : K \to \mathcal{F}(D)$ be a convex fuzzy mapping with weakly open fuzzy set-values and nonempty strong cut set $\{F_x \mid y \in D\}$ for an u.s.c. function $f : X \to [0, 1]$, $G : K \to \mathcal{G}(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values and nonempty cut set $\{G_x \mid y \in D\}$ for $\gamma \in (0, 1]$, a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{x \in C(x), x \in K$, be closed, and $M : K \times D \to 2^L(X, x)$ be a multivalued mapping. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(y, y)$ continuous and $H : K \times K \to 2^Z$ be $P$-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

(i) $\max(M(y, s), \eta(y, z)) \to \max(M(y, s), \eta(y, z))$ provided that $y \to y, s \to s$ and $z \to z$.

(ii) $\langle M(x, z), \eta(x, z) \rangle = 0$ and $H(x, z) = \{0\}$ for all $x \in K$.

(iii) there is a nonempty compact subset $B$ of $K$ such that for any nonempty finite subset $N$ of $K$, there is a nonempty compact convex subset $L_N$ of $K$ containing $N$ such that for any $y \in L_N \setminus B$, there exist $x \in L_N, z \in (G_x)_{\gamma} \gamma$ and $u \in H(x, y)$ such that

$$
\max(M(y, s), \eta(y, z)) + u \in -\ker C(y)
$$

for any $s \in (F_y)_{\beta}(y)$.

Then the problem (F-WVQLVI) is solvable, i.e., there exist $\bar{x} \in K$ and $\bar{s} \in (F_y)_{\beta}(y)$ such that

$$
\max(M(\bar{x}, s), \eta(x, z)) + u \notin -\ker C(x)
$$

for any $x \in K, z \in (G_x)_{\gamma}$ and $u \in H(x, \bar{x})$.

Proof. Define two multivalued mappings $\bar{F} : K \to 2^D$ and $\bar{G} : K \to 2^K$ by $\bar{F}(x) = \bar{F}(x)_{\beta}(x)$ and $\bar{G}(x) = (G_x)_{\gamma}$, for $x \in K$, respectively. It follows from Lemma 2.3. that $\bar{G}$ is l.s.c. and has nonempty convex-values and from Lemma 2.4, that $\bar{F}$ has nonempty convex-values such that $\bar{F}^{-1}(y)$ is open in $X$ for $y \in D$. Thus, by Lemma 3.1, there exists a continuous function $f : K \to D$ such that $f(x) \in \bar{F}(x)$ for $x \in K$. So, by Proposition 3.1, there exists $\bar{x} \in K$ such that

$$
\max(M(\bar{x}, f(\bar{x})), \eta(x, z)) + u \notin -\ker C(x)
$$

for any $x \in K, z \in (G_x)_{\gamma}$ and $u \in H(x, \bar{x})$. Letting $\bar{s} = f(\bar{x})$, we obtain the desired conclusion of Theorem 3.1.

From Theorem 3.1, we obtain the following theorem for Stampacchia type of the generalized weak vector quasivariational-like inequalities (WVQLVI) for multivalued mappings.

Theorem 3.2. Let $K$ be a nonempty paracompact convex subset of $X$ and $D$ be a nonempty closed convex subset of $Y$. Let $F : K \to \mathcal{F}(D)$ be a convex fuzzy mapping with nonempty convex-values and open lower sections, $G : K \to 2^K$ be a multivalued l.s.c. mapping with nonempty convex-values, a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{x \in C(x), x \in K$, be closed, and $M : K \times D \to 2^L(X, x)$ be a multivalued mapping. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(y, y)$ be continuous and $H : K \times K \to 2^Z$ be $P$-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

(i) $\max(M(y, s), \eta(y, z)) \to \max(M(y, s), \eta(y, z))$ provided that $y \to y, s \to s$ and $z \to z$.

(ii) $\langle M(x, z), \eta(x, z) \rangle = 0$ and $H(x, z) = \{0\}$ for all $x \in K$.

(iii) there is a nonempty compact subset $B$ of $K$ such that for any nonempty finite subset $N$ of $K$, there is a nonempty compact convex subset $L_N$ of $K$ containing $N$ such that, for any $y \in L_N \setminus B$, there exist $x \in L_N, z \in G(y)$ and $u \in H(x, y)$ such that

$$
\max(M(y, s), \eta(x, z)) + u \notin -\ker C(y), s \in F(y).
$$

Then the problem (WVQLVI) is solvable.

For the compact set case, by using Ky Fan's section theorem [7], we obtain the following existence of solutions for the vector variational inequalities (F-VQVI), (F-VVI), (VQVI) as special cases of (WVQLVI).

Theorem 3.3. ([8]) Let $K$ be a nonempty compact convex subset of $X$ and $D$ be a nonempty closed convex subset of $Y$. Let $F : K \to \mathcal{F}(D)$ be a convex fuzzy mapping with closed fuzzy set-values, $G : K \to \mathcal{G}(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values, $M : K \times D \to 2^L(X, x)$ be a multivalued mapping and a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{x \in C(x), x \in K$, be closed. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(y, y)$ be continuous, and $H : K \times K \to 2^Z$ be $P$-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

(i) there exist a l.s.c. function $\beta : X \to [0, 1]$ and a constant $\gamma \in (0, 1]$, such that for any $x \in K$, the cut sets $\{F_x \mid y \in D\}$ and $\{G_x \mid y \in D\}$ are nonempty;
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(i) \( \bigcup_{x \in K} (F_x)_{\beta(x)} \) is contained in some compact subset of \( D \);

(ii) \( \max \{ M(y_{\alpha}, s_{\alpha}), \eta(x, z_{\alpha}) \} \) converges to \( \max \{ M(y, s), \eta(x, z) \} \) provided that \( y_{\alpha} \to y, s_{\alpha} \to s \) and \( z_{\alpha} \to z \);

(iii) \( \langle M(x, \cdot), \eta(x, \cdot) \rangle = 0 \) and \( H(x, x) = \{0\} \) for all \( x \in K \).

Then the problem (F-VQVLI) is solvable from Theorem 2.1.

Corollary 3.4. (13) Let \( K \) be a nonempty compact convex subset of \( X \). Let \( F : K \to \mathcal{Y}(L(X, Y)) \) be a fuzzy mapping with closed fuzzy set-values, \( G : K \to \mathcal{Y}(K) \) be a convex fuzzy mapping with topologically open fuzzy set-values and a multivalued mapping \( W : K \to 2^Y \) defined by \( W(x) = Y \setminus \{ -\text{int} C(x) \} \), \( x \in K \), be closed, where \( \{ C(x) : x \in K \} \) is a family of solid convex cones in \( Y \).

Let \( P = \bigcap_{x \in K} C(x) \) and \( h : K \to Y \) be a continuous \( P \)-convex function. Suppose further that

(i) there exist a l.s.c. function \( \beta : X \to (0, 1] \) and a constant \( \gamma \in (0, 1] \) such that for any \( x \in K \) the cut sets \( (F_x)_{\beta(x)} \) and \( (G_x)_{\gamma} \) are nonempty;

(ii) \( \bigcup_{x \in K} (F_x)_{\beta(x)} \) is contained in some compact subset of \( L(X, Y) \);

(iii) for any \( x \in K \), there exists \( s \in (F_x)_{\beta(x)} \) such that \( (s, x - z) \notin -\text{int} C(x) \) for any \( z \in (G_x)_{\gamma} \).

Then the following variational inequality:

(F-VVI) Find \( \bar{x} \in K \) such that, for any \( x \in K \), there exists \( \bar{s} \in (F_x)_{\beta(x)} \) such that

\[ \langle \bar{s}, x - z \rangle + h(x) - h(\bar{x}) \notin -\text{int} C(x), \quad z \in (G_x)_{\gamma}, \]

is solvable.

The following theorem for the existence of solutions for (VQVLI) is a special case of Theorem 3.3.

Corollary 3.5. (18) Let \( K \) be a nonempty compact convex subset of \( X \) and \( D \) be a nonempty subset of \( Y \). Let \( F : K \to 2^D \) be closed, \( G : K \to 2^K \) be l.s.c. and nonempty convex-valued, \( M : K \times D \to 2^{L(x, Z)} \) be nonempty compact-valued and a multivalued mapping \( W : K \to 2^Z \) defined by \( W(x) = Z \setminus \{ -\text{int} C(x) \}, x \in K \), be closed. Let \( \eta : X \times X \to X \) be linear and \( H : K \times K \to 2^Z \) be \( P \)-convex with respect to the first variable and l.s.c. with respect to the second, where \( P := \bigcap_{x \in K} C(x) \).

Suppose further that

(i) \( \langle M(x, \cdot), \eta(x, \cdot) \rangle = 0 \) and \( H(x, x) = \{0\} \) for all \( x \in K \);

(ii) \( P \) is compact;

(iii) \( \max \{ M(y_{\alpha}, s_{\alpha}), \eta(x, z_{\alpha}) \} \) converges to \( \max \{ M(y, s), \eta(x, z) \} \) provided that \( y_{\alpha} \to y, s_{\alpha} \to s \) and \( z_{\alpha} \to z \).

Then the problem (VQVLI) is solvable.

Remark 3.1. Corollary 3.5 is a generalization of many outcomes in [1-5, 8, 10, 11, 14] and theirin.

References


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