

Ternary Distributive Structures and Quandles

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ABSTRACT. We introduce a notion of ternary distributive algebraic structure, give examples, and relate it to the notion of a quandle. Classification is given for low order structures of this type. Constructions of such structures from 3-Lie algebras are provided. We also describe ternary distributive algebraic structures coming from groups and give examples from vector spaces whose bases are elements of a finite ternary distributive set. We introduce a cohomology theory that is analogous to Hochschild cohomology and relate it to a formal deformation theory of these structures.

1. Introduction

Ternary operations, which are natural generalizations of binary operations, appear in many areas of mathematics and physics. An example of a ternary operation of an associative type is a map μ on a set X satisfying $\mu(\mu(x, y, z), u, v) = \mu(x, \mu(y, z, u), v) = \mu(x, y, \mu(z, u, v))$. Algebras with these multiplications are called totally associative ternary algebras and have been considered, for example, in [3, 4]. The first ternary algebraic structure given in an axiomatic form appeared in 1949 in the work of N. Jacobson [24]. He considered a Lie bracket $[x, y]$ in a Lie algebra \mathcal{L} and a subspace that is closed with respect to $[[x, y], z]$ which he called a Lie triple system. Since then, many works were devoted to ternary structures and their cohomologies (see for example [22, 10, 26, 27, 34, 36]). A typical example of an associative triple system is the ternary algebra of rectangular matrices introduced by M. R. Hestenes [23] where the ternary product is AB^*C (the $*$ stands for the

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conjugate transpose). In theoretical physics, the progress of quantum mechanics, Nambu mechanics, and the work of S. Okubo [31, 32] allowed an important development in the theory of ternary algebras (see [1, 2, 5] for example). Furthermore, this generalization of Hamiltonian systems by Nambu generated some profound studies of Nambu-Lie ternary algebras, which are generalizations of Lie algebras. The algebraic formulation of this structure was achieved by Fillipov [19] and Takhtajan [35] based on some generalization of the Jacobi identity.

Distributivity in algebraic structures appeared in many contexts, such as in the quasigroup theory, the semigroup theory, and the algebraic knot theory. The notion of a quandle (involutive quandle) appeared first as an abstraction of the notion of symmetric transformation, while the racks were studied in the context of the conjugation operation in a group. Around 1982, Joyce and Matveev introduced independently the notion of a quandle. They associated a quandle to each oriented knot called the knot quandle. Since then quandles and racks have been investigated by topologists in order to construct knot and link invariants and their higher analogues. We aim in this paper to extend these notions to ternary operations. This paper is organized as follows. In Section 2, we will review the basics on quandles and introduce ternary distributive structures and more generally n -ary distributive operations. We will define naturally the notion of ternary quandles and racks. We will provide some examples and describe some properties. In section 3, we will give the classification of ternary quandles of low order up to isomorphisms and we describe ternary distributive algebraic structures coming from groups. In Section 4, we will provide some constructions from ternary bialgebras and 3-Lie algebras. In Section 5 we will describe a low dimensional cohomology theory of ternary distributive bialgebra that fits with a deformation theory of ternary distributive operations. Section 6 is dedicated to a deformation theory of a weak ternary distributive bialgebra and in particular ternary quandles. In the appendix we give a classification of ternary distributive linear maps that are compatible with comultiplication.

2. Quandles and Ternary Distributive Structures

We begin this section by reviewing the basics of quandles and give examples in order to introduce their analogues in the ternary setting.

A *quandle*, X , is a set with a binary operation $(a, b) \mapsto a * b$ such that

- (i) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.
- (ii) For any $a, b \in X$, there is a unique $x \in X$ such that $a = x * b$.
- (iii) For any $a \in X$, $a * a = a$.

Axiom (ii) states that for each $u \in X$, the map $R_u : X \rightarrow X$ with $R_u(x) := x * u$ (right multiplication by u) is a bijection. A *rack* is a set with a binary operation that satisfies (i) and (ii).

Racks and quandles have been studied in, for example, [18, 25, 28]. The axioms of a quandle (i), (ii) and (iii) above correspond respectively to the Reidemeister moves

of type III, II, and I. For more details, see [18], for example. Here are some typical examples of quandles.

- Any set X with the operation $x * y = x$ for any $x, y \in X$ is a quandle called the *trivial* quandle. The trivial quandle of n elements is denoted by T_n .
- A group $X = G$ with n -fold conjugation as the quandle operation: $a * b = b^n a b^{-n}$.
- Let n be a positive integer. For elements $i, j \in \mathbb{Z}_n$ (integers modulo n), define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n . This set can be identified with the set of reflections of a regular n -gon with conjugation as the quandle operation.
- Any $\Lambda (= \mathbb{Z}[t, t^{-1}])$ -module M is a quandle with $a * b = ta + (1-t)b$, $a, b \in M$, called an *Alexander quandle*. Furthermore for a positive integer n , a *mod- n Alexander quandle* $\mathbb{Z}_n[t, t^{-1}]/(h(t))$ is a quandle for a Laurent polynomial $h(t)$. The mod- n Alexander quandle is finite if the coefficients of the highest and lowest degree terms of h are units in \mathbb{Z}_n .

Now we introduce the analogous notion of a quandle in the ternary setting.

Definition 2.1. Let Q be a set and $T : Q \times Q \times Q \rightarrow Q$ be a ternary operation on Q . The operation T is said to be *right distributive* if it satisfies the following condition for all $x, y, z, u, v \in Q$

$$(2.1) \quad T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v)). \quad (\text{right distributivity})$$

Remark 2.2. Note that one can similarly define the notions of left distributive as well as middle distributive. Through the rest of this paper, we will use distributive to refer to specifically right distributive.

Using the diagonal map $D : Q \rightarrow Q \times Q \times Q = Q^{\times 3}$ such that $D(x) = (x, x, x)$, equation (2.1) can be written, as a map from $Q^{\times 5}$ to Q , in the following form

$$(2.2) \quad T \circ (T \times id \times id) = T \circ (T \times T \times T) \circ \rho \circ (id \times id \times id \times D \times D),$$

where id stands for the identity map and in the whole paper we denote by $\rho : Q^{\times 9} \rightarrow Q^{\times 9}$ the map defined as $\rho = p_{6,8} \circ p_{3,7} \circ p_{2,4}$ where $p_{i,j}$ is the transposition i^{th} and j^{th} elements, i.e.

$$(2.3) \quad \rho(x_1, \dots, x_9) = (x_1, x_4, x_7, x_2, x_5, x_8, x_3, x_6, x_9).$$

Equation (2.2) as a new form of equation (2.1) will be used in section in the context of ternary bialgebras.

Definition 2.3. Let $T : Q \times Q \times Q \rightarrow Q$ be a ternary operation on a set Q . The pair (Q, T) is said to be *ternary shelf* if T satisfies identity (2.1). If, in addition, for all $a, b \in Q$, the map $R_{a,b} : Q \rightarrow Q$ given by $R_{a,b}(x) = T(x, a, b)$ is invertible,

then (Q, T) is said to be *ternary rack*. If further T satisfies $T(x, x, x) = x$, for all $x \in Q$, then (Q, T) is called a *ternary quandle*.

The figure below is a diagrammatic representation of equation (1).

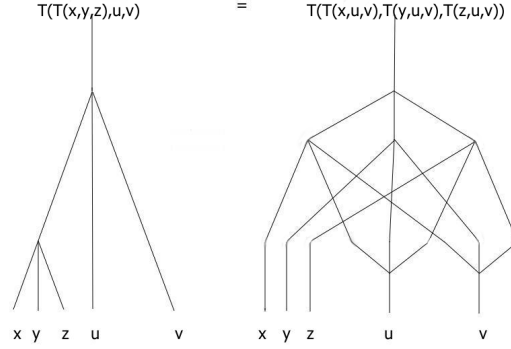


Fig. 1: Ternary Distributive Condition

Example 2.4. Let $(Q, *)$ be a quandle and define a ternary operation on Q by $T(x, y, z) = (x * y) * z, \forall x, y, z \in Q$. It is straightforward to see that (Q, T) is a ternary quandle. Note that in this case $R_{a,b} = R_b \circ R_a$. We will say that this ternary quandle is induced by a (binary) quandle.

Example 2.5. Let $(M, *)$ be an Alexander quandle, then the ternary quandle coming from M has the operation $T(x, y, z) = t^2x + t(1-t)y + (1-t)z$.

Example 2.6. Let M be any Λ -module where $\Lambda = \mathbb{Z}[t^{\pm 1}, s]$. The operation $T(x, y, z) = tx + sy + (1-t-s)z$ defines a ternary quandle structure on M . We call this an *affine* ternary quandle.

Example 2.7. Consider \mathbb{Z}_8 with the ternary operation $T(x, y, z) = 3x + 2y + 4z$. This affine ternary quandle is not induced by an Alexander quandle structure since 3 is not a square in \mathbb{Z}_8 .

Example 2.8. Any group G with the ternary operation $T(x, y, z) = xy^{-1}z$ gives an example of ternary quandle. This is called *heap* (sometimes also called a *groud*) of the group G .

A morphism of ternary quandles is a map $\phi : (Q, T) \rightarrow (Q', T')$ such that

$$\phi(T(x, y, z)) = T'(\phi(x), \phi(y), \phi(z)).$$

A bijective ternary quandle endomorphism is called ternary quandle automorphism. Therefore, we have a category whose objects are ternary quandles and morphisms as defined above.

Definition 2.9. A ternary rack (resp. ternary quandle) (Q, T) is said to be pointed if there is a distinguished element denoted $1 \in Q$ such that, for all $x, y \in Q$, $T(x, 1, 1) = x$, and $T(1, x, y) = 1$.

As in the case of the binary quandle there is a notion of *medial* ternary quandle.

Definition 2.10.([6]) A ternary quandle (Q, T) is said to be *medial* if for all $a, b, c, d, e, f, g, h, k \in Q$, the following identity is satisfied

$$T(T(a, b, c), T(d, e, f), T(g, h, k)) = T(T(a, d, g), T(b, e, h), T(c, f, k)).$$

This definition of *mediality* can be written in term of the following commutative diagram

$$\begin{array}{ccc}
 & \underbrace{Q \times \cdots \times Q}_{9 \text{ times}} & \\
 & \swarrow T \times T \times T & \searrow \rho \\
 Q \times Q \times Q & & \underbrace{Q \times \cdots \times Q}_{9 \text{ times}} \\
 \downarrow T & & \downarrow T \times T \times T \\
 Q & \xleftarrow{T} & Q \times Q \times Q
 \end{array}$$

Where $\rho = (24)(37)(68)$ is the permutation of the set $\{1, \dots, 9\}$ defined above.

Example 2.11. Every affine ternary quandle is medial.

We generalize the notion of ternary quandle to n -ary setting.

Definition 2.12. An n -ary distributive set is a pair (Q, T) where Q is a set and $T : Q^{\times n} \rightarrow Q$ is an n -ary operation satisfying the following conditions:

1.

$$\begin{aligned}
 & T(T(x_1, \dots, x_n), u_1, \dots, u_{n-1}) = \\
 & T(T(x_1, u_1, \dots, u_{n-1}), T(x_2, u_1, \dots, u_{n-1}), \dots, T(x_n, u_1, \dots, u_{n-1})),
 \end{aligned}$$

$\forall x_i, u_i \in Q$ (distributivity).

2. For all $a_1, \dots, a_{n-1} \in Q$, the map $R_{a_1, \dots, a_{n-1}} : Q \rightarrow Q$ given by

$$R_{a_1, \dots, a_{n-1}}(x) = T(x, a_1, \dots, a_{n-1})$$

is invertible.

3. For all $x \in Q$,

$$T(x, \dots, x) = x.$$

If T satisfies only condition(1), then (Q, T) is said to be an n -ary shelf. If both conditions (1) and (2) are satisfied then (Q, T) is said to be an n -ary rack. If all three conditions (1), (2) and (3) are satisfied then (Q, T) is said to be an n -ary quandle.

Definition 2.13. An n -ary quandle (Q, T) is said to be *medial* if for all $x_{ij} \in Q, 1 \leq i, j \leq n$, the following identity is satisfied

$$T(T(x_{11}, x_{12}, \dots, x_{1n}), T(x_{21}, x_{22}, \dots, x_{2n}), \dots, T(x_{n1}, x_{n2}, \dots, x_{nn})) = T(T(x_{11}, x_{21}, \dots, x_{n1}), T(x_{12}, x_{22}, \dots, x_{n2}), \dots, T(x_{1n}, x_{2n}, \dots, x_{nn})).$$

3. Classification of Ternary Quandles of Low Orders

We give in this section the classification of ternary quandles up to isomorphisms. We provide all ternary quandles of order 2 and 3. Moreover we describe ternary distributive structures coming from groups. Recall that a ternary quandle is a pair (Q, T) , where Q is a set and T a ternary operation, satisfying the following conditions

(I) $T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$, for all $x, y, z, u, v \in Q$,

(II) for all $a, b \in Q$, the map $R_{a,b} : Q \rightarrow Q$ given by $R_{a,b}(x) = T(x, a, b)$ is invertible,

(III) for all $x \in Q$, $T(x, x, x) = x$.

3.1 Ternary quandles of order two

We have the following lemma which states that there are two non-isomorphic ternary quandle structures on a set of two elements.

Lemma 3.1. *In size two, all ternary quandles are affine, and are divided into two isomorphism classes, represented by the trivial ternary quandle, and the one with $T(x, y, z) = x + y + z \pmod{2}$.*

Proof. Let $Q = \{1, 2\}$ and T a ternary quandle operation on Q . Then we have $T(1, 1, 1) = 1$ and $T(2, 1, 1) = 2$. Similarly, we have $T(2, 2, 2) = 2$ and $T(1, 2, 2) = 1$. Now we need to choose a value for $T(1, 1, 2)$. We distinguish two cases:

Case 1: Assume $T(1, 1, 2) = 1$, this implies $T(2, 1, 2) = 2$ (by second axiom). We claim that in this case $T(1, 2, 1)$ can not equal 2, otherwise $T(2, 2, 1) = 1$ (again

axiom (II)). Now use axiom (I) of right-self-distributivity to get $T(T(2, 1, 2), 2, 1) = T(T(2, 2, 1), T(1, 2, 1), T(2, 2, 1))$ implying that $T(2, 2, 1) = T(1, 2, 1)$ but this contradicts the bijectivity of axiom (II). Then $T(1, 2, 1) = 1$ and $T(2, 2, 1) = 2$. This ends the proof.

Case 2: Assume $T(1, 1, 2) = 2$, this implies $T(2, 1, 2) = 1$ (by second axiom). As in case 1, we prove similarly that $T(1, 2, 1)$ can not equal 1, thus $T(1, 2, 1) = 2$ and $T(2, 2, 1) = 1$. Now, the only non-trivial bijection of the set $\{1, 2\}$ is the transposition sending 1 to 2. It's easy to see that this transposition is not a homomorphism between the two ternary quandles given in case 1 and case 2. \square

3.2 Ternary quandles of order three

To help classify the ternary quandles two observations are useful. First we note that every ternary quandle is related to some (binary) quandle.

Remark 3.2. If (Q, T) is a ternary quandle, then $(Q, *)$, where $x * y = T(x, y, y)$ is a (binary) quandle.

We shall refer to this related quandle as the *associated quandle*. We now consider how the relation between associated quandles extends to ternary quandles.

Lemma 3.3. *Let (Q, T) be a ternary quandle, and $(Q, *)$, be the associated quandle defined by $x * y = T(x, y, y)$. If $(R, *')$ is a quandle such that $(Q, *) \cong (R, *')$, then there exists a ternary quandle (R, T') $\cong (Q, T)$ such that $x *' y = T'(x, y, y)$.*

Proof. This is easily shown by setting $T'(x, y, z) = \phi(T(\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(z)))$ where $\phi : Q \rightarrow R$ is an isomorphism from $(Q, *)$ to $(R, *')$. \square

With these facts we now see that we may limit the task of generating isomorphically distinct ternary quandles by generating them based on isomorphically distinct quandles. We developed a simple program using the conditions defining a ternary quandle to compute all ternary quandles of order 3. The results of which we used to obtain the following

Lemma 3.4. *There are 31 isomorphically distinct ternary quandles of order 3. Six of these are affine: the trivial ternary quandle T_0 , as well as two more with trivial associated quandle, T_{14} defined by $T(x, y, z) = x + y + 2z \pmod{3}$, and T_{15} defined by $T(x, y, z) = x + 2y + z$, as well as three with the connected associated quandle, T_1 defined by $T(x, y, z) = 2x + 2z$, T_2 defined by $T(x, y, z) = 2x + y + z$, and T_5 defined by $T = 2x + 2y$.*

Additionally we found that 14 were connected, including the non-trivial affine structures, as well as the remaining structures with the connected quandle as their associated quandle and 6 with the trivial associated quandle $T_6, T_7, T_{10}, T_{12}, T_{13}$ and T_{16} .

Since for each fixed a, b , the map $x \mapsto T(x, a, b)$ is a permutation, then in the following table we describe all ternary quandles of order three in terms of the columns of the Cayley table. Each column is a permutation of the elements and is

z=1			z=2			z=3		
1	2	3	2	1	1	3	1	1
2	1	2	1	2	3	2	3	2
3	3	1	3	3	2	1	2	3

Table 1: Cayley representation of ternary quandle T_{12}

T	z=1	z=2	z=3
T_{12}	(1),(12),(13)	(12),(1),(23)	(13),(23),(1)

Table 2: Permutation representation of ternary quandle T_{12}

described in standard notation that is by explicitly writing it in terms of products of disjoint cycles. Thus for a given z we give the permutations resulting from fixing $y = 1, 2, 3$. For example, the ternary set $T_{12}(x, y, z)$ with the Cayley Table 1 will be represented with the permutations (1), (12), (13); (12), (1), (23); (13), (23), (1). This will appear on Table 3 as shown in Table 2.

Additionally we organize the table based on the associated quandle, given in similar permutation notation.

3.3 Ternary distributive structures from groups

We search for ternary distributive structures coming from groups. We have the following necessary condition.

Lemma 3.5. *Let x, y, z be three fixed elements in a group G . Let $w(x, y, z) = a_1^{e_1} a_2^{e_2} \dots a_n^{e_n}$ such that $a_i \in \{x, y, z\}$ and $e_i = \pm 1$. If w is defined such that (I) $\sum_{i=1}^n e_i = 1$, (II) there exists a unique i such that $a_i = x$, and (III) $w(x, y, z)$ satisfies equation (2.1) of Definition , then w defines a ternary quandle over the group G .*

The condition $\sum_{i=1}^n e_i = 1$ is a result of axiom (III) and the condition $\sum_{i \in I} e_i = \pm 1$ is a result of axiom (II).

Using the sufficient conditions, we found three families of group words defining ternary quandles over G . Words of the form $x(a^{-1}b)^n$, $(ab^{-1})^n x$ and wxw^{-1} , where $a, b \in \{y, z\}$, and w is any word over $\{y, z\}$, clearly satisfy the first and second condition and by reducing the words $w(w(x, y, z), u, v)$ and $w(w(x, u, v), w(y, u, v), w(z, u, v))$, we easily obtain the third condition.

Remark 3.6. In [29], the author investigated some ternary operations coming from coloring the four regions around crossings of classical knot diagrams.

Ternary Distributive Sets With Associated Quandle (1),(1),(1)							
T	$z=1$	$z=2$	$z=3$	T	$z=1$	$z=2$	$z=3$
T_0	(1),(1),(1)	(1),(1),(1)	(1),(1),(1)	T_1	(1),(1),(1)	(1),(1),(1)	(12),(12),(1)
T_2	(1),(1),(1)	(1),(1),(23)	(1),(23),(1)	T_3	(1),(1),(1)	(23),(1),(1)	(23),(1),(1)
T_4	(1),(1),(1)	(23),(1),(23)	(23),(23),(1)	T_5	(1),(1),(12)	(1),(1),(12)	(12),(12),(1)
T_6	(1),(1),(123)	(123),(1),(1)	(1),(123),(1)	T_7	(1),(1),(132)	(132),(1),(1)	(1),(132),(1)
T_8	(1),(1),(13)	(13),(1),(13)	(13),(1),(1)	T_9	(1),(23),(23)	(23),(1),(23)	(23),(23),(1)
T_{10}	(1),(23),(23)	(13),(1),(13)	(12),(12),(1)	T_{11}	(1),(12),(12)	(12),(1),(12)	(1),(1),(1)
T_{12}	(1),(12),(13)	(12),(1),(23)	(13),(23),(1)	T_{13}	(1),(123),(123)	(123),(1),(123)	(123),(123),(1)
T_{14}	(1),(123),(132)	(132),(1),(123)	(123),(132),(1)	T_{15}	(1),(132),(123)	(123),(1),(132)	(132),(123),(1)
T_{16}	(1),(13),(12)	(23),(1),(12)	(23),(13),(1)				
Ternary Distributive Sets With Associated Quandle (1),(1),(12)							
T	$z=1$	$z=2$	$z=3$	T	$z=1$	$z=2$	$z=3$
T_0	(1),(1),(1)	(1),(1),(1)	(1),(1),(12)	T_1	(1),(1),(1)	(1),(1),(1)	(12),(12),(12)
T_2	(1),(1),(12)	(1),(1),(12)	(1),(1),(12)	T_3	(1),(1),(12)	(1),(1),(12)	(12),(12),(12)
T_4	(1),(12),(1)	(12),(1),(1)	(1),(1),(12)	T_5	(1),(12),(1)	(12),(1),(1)	(12),(12),(12)
T_6	(1),(12),(12)	(12),(1),(12)	(1),(1),(12)	T_7	(1),(12),(12)	(12),(1),(12)	(12),(12),(12)
Ternary Distributive Sets With Associated Quandle (23),(13),(12)							
T	$z=1$	$z=2$	$z=3$	T	$z=1$	$z=2$	$z=3$
T_0	(23),(1),(1)	(1),(13),(1)	(1),(1),(12)	T_1	(23),(23),(23)	(13),(13),(13)	(12),(12),(12)
T_2	(23),(12),(13)	(12),(13),(23)	(13),(23),(12)	T_3	(23),(123),(132)	(132),(13),(123)	(123),(132),(12)
T_4	(23),(132),(123)	(123),(13),(132)	(132),(123),(12)	T_5	(23),(13),(12)	(23),(13),(12)	(23),(13),(12)

Table 3: Isomorphism classes of ternary quandles of order 3

We mention that the axioms satisfied by his ternary operations involve only four arguments while our ternary distributive operation axiom involves five arguments (see equation (2.1)). Even though, the ternary operation of the heap of a group $T(x, y, z) = xy^{-1}z$ happened to be an example for both his operations and ours, the difference is that any permutation of the three letters x, y, z in this operation is also an example in his context, while in our situation the ternary operation obtained by the transposition of x and y that is $T(x, y, z) = yx^{-1}z$ is not distributive operation. This shows that his ternary operations and ours are different.

4. Constructions from Ternary Bialgebras and 3-Lie Algebras

We provide in this section some constructions of ternary shelves involving ternary bialgebra structures and 3-Lie algebras.

4.1 Ternary bialgebras

We start by recalling definitions of ternary algebras, coalgebras and bialgebras. See [6, 16, 21, 37] for references about ternary bialgebras.

Definition 4.1. A ternary \mathbb{K} -algebra is a triple (A, μ, η) where A is a vector space over a field \mathbb{K} with a multiplication $\mu : A \otimes A \otimes A \rightarrow A$ and a unit $\eta : \mathbb{K} \rightarrow A$ that are linear maps such that the following associativity identity is satisfied

$$(4.1) \quad \mu \circ (\mu \otimes id \otimes id) = \mu \circ (id \otimes \mu \otimes id) = \mu \circ (id \otimes id \otimes \mu)$$

and the following property of the unit is also satisfied

$$\mu \circ (\eta \otimes \eta \otimes id) = \mu \circ (\eta \otimes id \otimes \eta) = \mu \circ (id \otimes \eta \otimes \eta) = id.$$

The triple (A, μ, η) defines a *weak* ternary \mathbb{K} -algebra if, instead of identity (4.1), the following weak associativity identity holds

$$(4.2) \quad \mu \circ (\mu \otimes id \otimes id) = \mu \circ (id \otimes id \otimes \mu)$$

Ternary coalgebras are defined similarly by changing the directions of the arrows in the previous definition. Precisely,

Definition 4.2. A vector space A is a ternary \mathbb{K} -coalgebra if it has a coalgebra comultiplication Δ that is a linear map $\Delta : A \rightarrow A \otimes A \otimes A$ satisfying the following coassociativity identity:

$$(\Delta \otimes id \otimes id) \circ \Delta = (id \otimes \Delta \otimes id) \circ \Delta = (id \otimes id \otimes \Delta) \circ \Delta = id.$$

The ternary coalgebra is said to be counital if there exists a map $\varepsilon : A \rightarrow \mathbb{K}$ such that

$$(4.3) \quad (\varepsilon \otimes id \otimes id) \circ \Delta = (id \otimes \varepsilon \otimes id) \circ \Delta = (id \otimes id \otimes \varepsilon) \circ \Delta.$$

The triple (A, Δ, ε) defines a ternary weak \mathbb{K} -coalgebra if, instead of identity (4.3), the following weak coassociativity identity holds

$$(4.4) \quad (\Delta \otimes id \otimes id) \circ \Delta = (id \otimes id \otimes \Delta) \circ \Delta.$$

A linear map $f : A \rightarrow A$ is called compatible with a comultiplication and the counit ε if

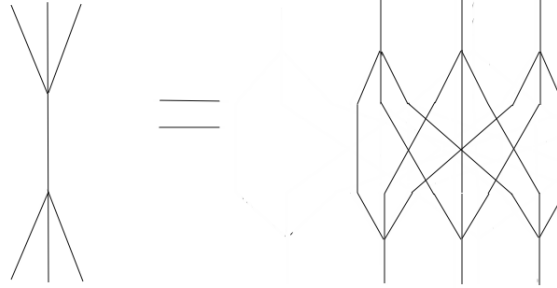
$$\Delta \circ f = (f \otimes f \otimes f) \circ \Delta \quad \text{and} \quad \varepsilon f = \varepsilon$$

Definition 4.3. We say that a ternary operation $\mu : A \otimes A \otimes A \rightarrow A$ is compatible with a comultiplication Δ if and only if

$$(4.5) \quad \Delta \circ \mu = (\mu \otimes \mu \otimes \mu) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta),$$

where $\rho : A^{\otimes 9} \rightarrow A^{\otimes 9}$ is defined in equation (2.3).

The following is a figure of compatibility of ternary operation with comultiplication



Compatibility of Ternary and comultiplication operations

Definition 4.4. A ternary bialgebra is a quintuplet $(A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) is a ternary algebra, (A, Δ, ϵ) is a ternary coalgebra and the multiplication μ and the unit η are coalgebra morphisms (equivalent to Δ and ϵ being algebra morphisms).

Example 4.5.

- Group algebras: Let G be a (multiplicative) group, and $\mathbb{K}[G]$ be the group algebra. Then $A = \mathbb{K}[G]$ becomes a non-unital weak ternary bialgebra with the multiplication defined as $T(g, h, k) = gh^{-1}k$, the comultiplication and counit are given respectively by $\Delta(g) = g \otimes g \otimes g$ and $\epsilon(g) = 1$. This ternary algebra is weak and not unital.
- Function algebra on groups: Let G be a finite group. Using $\mathbb{K}^{G \times G \times G} \cong \mathbb{K}^G \otimes \mathbb{K}^G \otimes \mathbb{K}^G$, the set \mathbb{K}^G of functions from G to \mathbb{K} has a ternary pointwise multiplication and a ternary comultiplication $\Delta : \mathbb{K}^G \rightarrow \mathbb{K}^{G \times G \times G}$ given by $\Delta(f)(u \otimes v \otimes w) = f(uvw)$. Now \mathbb{K}^G has basis (the characteristic function) $\delta_g : G \rightarrow \mathbb{K}$ defined by $\delta_g(x) = 1$ if $x = g$ and zero otherwise. We have
$$\Delta(\delta_h) = \sum_{uvw=h} \delta_u \otimes \delta_v \otimes \delta_w.$$

4.2 3-Lie algebras

In the following we recall the definition of a 3-Lie algebra and show how to derive a ternary distributive operation from it. For further properties and results about 3-Lie algebras, we refer the reader to [1, 2, 5, 15, 19, 31, 32].

Definition 4.6. A 3-Lie algebra is a \mathbb{K} -vector space L together with a skewsymmetric ternary operation $[\cdot, \cdot, \cdot]$ satisfying, for all $x_1, \dots, x_5 \in L$,

$$[[x_1, x_2, x_3], x_4, x_5] = [[x_1, x_4, x_5], x_2, x_3] + [x_1, [x_2, x_4, x_5], x_3] + [x_1, x_2, [x_3, x_4, x_5]].$$

Given a 3-Lie algebra L , we can construct a ternary coalgebra $N = \mathbb{K} \oplus L$ by setting for all $x \in L$,

$$(4.6) \quad \Delta(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \quad \Delta(1) = 1 \otimes 1 \otimes 1, \quad \varepsilon(1) = 1 \text{ and } \varepsilon(x) = 0.$$

We define a linear map $T : N \otimes N \otimes N \rightarrow N$, for all $x, y, z \in L$, by

$$\begin{aligned} T(x \otimes y \otimes z) &= [x, y, z], \quad T(1 \otimes 1 \otimes 1) = 1, \quad T(x \otimes 1 \otimes 1) = x, \\ T(x \otimes 1 \otimes y) &= T(1 \otimes x \otimes 1) = T(x \otimes y \otimes 1) = T(1 \otimes x \otimes y) = T(1 \otimes 1 \otimes x) = 0. \end{aligned}$$

That is for $a, b, c \in \mathbb{K}$

$$T((a+x) \otimes (b+y) \otimes (c+z)) = abc + bcx + [x, y, z] = (abc, bcx + [x, y, z]).$$

As in the previous situation we have

Proposition 4.7. *The map T defined above satisfies the ternary distributive condition (2.2).*

Proof. In a similar manner to the proof of the previous theorem, we write explicitly LHS and RHS,

$$LHS = T(T \otimes id \otimes id)((a + \sum a_x x) \otimes (b + \sum b_y y) \otimes (c + \sum c_z z) \otimes (d + \sum_u d_u u) \otimes (f + \sum_v f_v v)).$$

$$\begin{aligned} RHS &= T(T \otimes T \otimes T)\rho(id \otimes id \otimes id \otimes D \otimes D) \\ &((a + \sum a_x x) \otimes (b + \sum b_y y) \otimes (c + \sum c_z z) \otimes (d + \sum_u d_u u) \otimes (f + \sum_v f_v v)). \end{aligned}$$

By expanding the LHS and RHS and equating them we obtain the result. \square

A direct verification shows that this map T also satisfies the equation $\Delta T = (T \otimes T \otimes T)\rho(\Delta \otimes \Delta \otimes \Delta)$ giving the following

Proposition 4.8. *The map T defined above is a coalgebra homomorphism.*

4.3 Distributive ternary bialgebras

We introduce here a notion of ternary distributive bialgebra.

Definition 4.9. A ternary distributive bialgebra (resp. ternary distributive weak bialgebra) is a triple (A, T, Δ) , where A is a vector space, $T : A^{\otimes 3} \rightarrow A$ a ternary operation, $\Delta : A \rightarrow A^{\otimes 3}$ a ternary comultiplication, such that

1. the operation T is distributive, meaning it satisfies equation (2.2),
2. the comultiplication Δ is coassociative (resp. weak coassociative),
3. the maps T and Δ are compatible, that is

$$(4.7) \quad \Delta \circ T = (T \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta).$$

It turns out that any ternary operation T over a set X endows $\mathbb{K}[X]$ with a structure of ternary distributive bialgebra with a comultiplication D defined on the generators as $D(x) = x \otimes x \otimes x$ for all $x \in X$, and then extended linearly. Indeed, D is coassociative and compatible with T . We have for $x, y, z \in X$

$$\begin{aligned} (T \otimes T \otimes T) \circ \rho \circ (D \otimes D \otimes D)(x \otimes y \otimes z) \\ &= (T \otimes T \otimes T)(\rho(x \otimes x \otimes x \otimes y \otimes y \otimes y \otimes z \otimes z \otimes z)) \\ &= (T \otimes T \otimes T)(x \otimes y \otimes z \otimes x \otimes y \otimes z \otimes x \otimes y \otimes z) \\ &= D \circ T(x \otimes y \otimes z). \end{aligned}$$

We note that a distributive ternary bialgebra can be associated to any 3-Lie algebra as made clear in the previous section .

5. Differentials and Cohomology

In this section, we provide a cohomology theory of ternary distributive bialgebras that fits with a deformation theory of ternary distributive operations.

Let (Q, T) be a ternary distributive set, \mathbb{K} be an algebraically closed field of characteristic zero and let $A = \mathbb{K}[Q]$ be the vector space spanned by the elements of Q . We extend by bilinearity the ternary distributive operation T to A .

We define low dimensional cochain groups by

$$\mathcal{C}^1 := Hom(A, A), \quad \mathcal{C}^2 := Hom(A^{\otimes 3}, A) \oplus Hom(A, A^{\otimes 3}),$$

and

$$\mathcal{C}^3 := Hom(A^{\otimes 5}, A) \oplus Hom(A^{\otimes 3}, A^{\otimes 3}) \oplus Hom(A, A^{\otimes 5}).$$

First differentials: For $f \in \mathcal{C}^1$, we define

$$\delta_m^1(f) := T(f \otimes id \otimes id) + T(id \otimes f \otimes id) + T(id \otimes id \otimes f) - fT,$$

and

$$\delta_c^1(f) := (f \otimes id \otimes id)\Delta + (id \otimes f \otimes id)\Delta + (id \otimes id \otimes f)\Delta - \Delta f.$$

The differential \mathcal{D}^1 is given by

$$\mathcal{D}^1(f) := \delta_m^1(f) - \delta_c^1(f) : \mathcal{C}^1 = Hom(A, A) \rightarrow Hom(A^{\otimes 3}, A) \oplus Hom(A, A^{\otimes 3}) = \mathcal{C}^2.$$

Second differentials: we define the second differentials for $\psi_1 \in \mathcal{C}^1$ and $\psi_2 \in \mathcal{C}^2$ by

$$\begin{aligned}
d^{2,1}(\psi_1, \psi_2) &= [\psi_1(T \otimes id \otimes id) + T(\psi_1 \otimes id \otimes id)] - \\
&\quad [\psi_1(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) + \\
&\quad T(\psi_1 \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) + \\
&\quad T(T \otimes \psi_1 \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) + \\
&\quad T(T \otimes T \otimes \psi_1) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) + \\
&\quad T(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \psi_2 \otimes \Delta) + \\
&\quad T(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \psi_2)], \\
d^{2,2}(\psi_1, \psi_2) &= [\psi_2 T + \Delta \psi_1] - [(\psi_1 \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) + \\
&\quad (T \otimes \psi_1 \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) + \\
&\quad (T \otimes T \otimes \psi_1) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) + \\
&\quad (T \otimes T \otimes T) \circ \rho \circ (\psi_2 \otimes \Delta \otimes \Delta) + \\
&\quad (T \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \psi_2 \otimes \Delta) + \\
&\quad (T \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \psi_2)], \\
d^{2,3}(\psi_1, \psi_2) &= [(\psi_2 \otimes id \otimes id)\Delta + (\Delta \otimes id \otimes id)\psi_2] - \\
&\quad [(id \otimes id \otimes \psi_2)\Delta + (id \otimes id \otimes \Delta)\psi_2].
\end{aligned}$$

Remark that this last formula of $d^{2,3}(\psi_1, \psi_2)$ involves only ψ_2 and it corresponds to the co-Hochschild 2-differential of the ternary comultiplication. The differential \mathcal{D}^2 is given by

$$\mathcal{D}^2(\psi_1, \psi_2) := d^{2,1}(\psi_1, \psi_2) + d^{2,2}(\psi_1, \psi_2) + d^{2,3}(\psi_1, \psi_2).$$

Proposition 5.1. *The composite $\mathcal{D}^2 \circ \mathcal{D}^1$ is zero.*

Proof. By assumption we have $\mathcal{D}^1(f) = (\psi_1, \psi_2)$, where $\psi_1 = \delta_m^1(f)$ and $\psi_2 = \delta_c^1(f)$. First we prove that $d^{2,1}(\mathcal{D}^1(f)) = 0$, then exactly using the same techniques we obtain that $d^{2,2}(\mathcal{D}^1(f)) = 0$. Now the details:

We write $d^{2,1}(\mathcal{D}^1(f))$ using eight terms T_1, \dots, T_8 as follows

$$d^{2,1}(\mathcal{D}^1(f)) = [T_1 + T_2] - [T_3 + T_4 + T_5 + T_6 + T_7 + T_8],$$

Let $T_{i,j}$ represents the j -th term of T_i (in the order given). Via the ternary distributive equation (2.1) and simple cancellation, the terms can be shown to cancel (in pairs) as follows: $T_{1,1}$ & $T_{2,4}$, $T_{1,2}$ & $T_{7,4}$, $T_{1,3}$ & $T_{8,4}$, $T_{1,4}$ & $T_{3,4}$, $T_{2,1}$ & $T_{4,1}$, $T_{2,2}$ & $T_{5,1}$, $T_{2,3}$ & $T_{6,1}$, $T_{3,1}$ & $T_{4,4}$, $T_{3,2}$ & $T_{5,4}$, $T_{3,3}$ & $T_{6,4}$, $T_{4,2}$ & $T_{7,1}$, $T_{4,3}$ & $T_{8,1}$, $T_{5,2}$ & $T_{7,2}$, $T_{5,3}$ & $T_{8,2}$, $T_{6,2}$ & $T_{7,3}$, $T_{6,3}$ & $T_{8,4}$, so that we then obtain at the end $\mathcal{D}^2(\mathcal{D}^1(f)) = 0$. As we mentioned before using exactly the same techniques we obtain that $d^{2,2}(\mathcal{D}^1(f)) = 0$. Since $d^{2,3}$ corresponds to the coHochschild 2-differential for the ternary multiplication Δ , it's straightforward that $d^{2,3}(\mathcal{D}^1(f)) = 0$. Thus $\mathcal{D}^2 \circ \mathcal{D}^1$ is zero. \square

The 1-cocycle spaces of A are

$$Z_m^1(A, A) = \{f : A \rightarrow A : \delta_m^1 f = 0\}, \quad Z_c^1(A, A) = \{f : A \rightarrow A : \delta_c^1 f = 0\}$$

and

$$Z^1(A, A) = Z_m^1 \cap Z_c^1 = H^1(A, A).$$

This is the space of maps which are simultaneously derivations and coderivations.

The 2-coboundaries space of A is

$$B^2(A, A) = \text{Im}(\mathcal{D}^1).$$

The 2-cocycles space of A is

$$Z^2(A, A) = \text{ker}(\mathcal{D}^2).$$

Then the second cohomology group is given by the quotient $Z^2(A, A)/B^2(A, A)$.

6. One-Parameter Formal Deformations

In this section we extend to ternary distributive weak bialgebras the theory of deformation of rings and associative algebras introduced by Gerstenhaber [20] and by Nijenhuis and Richardson for Lie algebras [30]. The fundamental results of Gerstenhaber's theory connect deformation theory with the suitable cohomology groups. This theory was extended to ternary algebras of associative type in [3, 4].

In the following we define the concept of deformation for a ternary distributive bialgebra and provide the connection to cohomology groups. The idea is to deform both the ternary multiplication and the ternary comultiplication at the same time.

Let (A, T, Δ) be a ternary distributive bialgebra. A deformation of (A, T, Δ) is a $\mathbb{K}[[t]]$ -bialgebra (A_t, T_t, Δ_t) , where $A_t = A \otimes \mathbb{K}[[t]]$ and $A_t/(tA_t) \cong A$. Deformations of T and Δ are given by $T_t = T + tT_1 + \cdots + t^n T_n + \cdots : A_t \otimes A_t \otimes A_t \rightarrow A_t$ and $\Delta_t = \Delta + t\Delta_1 + \cdots + t^n \Delta_n + \cdots : A_t \rightarrow A_t \otimes A_t \otimes A_t$ where $T_i : A \otimes A \otimes A \rightarrow A$, $\Delta_i : A \rightarrow A \otimes A \otimes A$, $i = 1, 2, \dots$, are sequences of maps.

Suppose $\bar{T} = T + tT_1 + \cdots + t^n T_n$ and $\bar{\Delta} = \Delta + t\Delta_1 + \cdots + t^n \Delta_n$ satisfy the bialgebra conditions (distributivity, coassociativity and compatibility) mod t^{n+1} , and suppose that there exist $T_{n+1} : A \otimes A \otimes A \rightarrow A$ and $\Delta_{n+1} : A \rightarrow A \otimes A \otimes A$ such that $\bar{T} + t^{n+1}T_{n+1}$ and $\bar{\Delta} + t^{n+1}\Delta_{n+1}$ satisfy the bialgebra conditions mod t^{n+2} . Define $\phi_1 \in \text{Hom}(A^{\otimes 5}, A)$, $\phi_2 \in \text{Hom}(A^{\otimes 3}, A^{\otimes 3})$, and $\phi_3 \in \text{Hom}(A, A^{\otimes 5})$ by

$$(6.1) \quad \begin{aligned} \bar{T}(\bar{T} \otimes id \otimes id) - \bar{T}(\bar{T} \otimes \bar{T} \otimes \bar{T}) \circ \rho \circ (id \otimes id \otimes id \otimes \bar{\Delta} \otimes \bar{\Delta}) \\ = t^{n+1}\phi_1 \quad \text{mod } t^{n+2}, \end{aligned}$$

$$(6.2) \quad \begin{aligned} \bar{\Delta} \bar{T} - (\bar{T} \otimes \bar{T} \otimes \bar{T}) \circ \rho \circ (\bar{\Delta} \otimes \bar{\Delta} \otimes \bar{\Delta}) \\ = t^{n+1}\phi_2 \quad \text{mod } t^{n+2}, \end{aligned}$$

$$(6.3) \quad \begin{aligned} (\bar{\Delta} \otimes id \otimes id)\bar{\Delta} - (id \otimes id \otimes \bar{\Delta})\bar{\Delta} \\ = t^{n+1}\phi_3 \quad \text{mod } t^{n+2}. \end{aligned}$$

Now expanding these three equations we obtain the values of ϕ_1, ϕ_2 and ϕ_3 for the case of $n = 0$:

$$\begin{aligned} \phi_1 = & [T_{n+1}(T \otimes id \otimes id) + T(T_{n+1} \otimes id \otimes id)] \\ & - [T_{n+1}(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) \\ & + T(T_{n+1} \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) \\ & + T(T \otimes T_{n+1} \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) \\ & + T(T \otimes T \otimes T_{n+1}) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta) \\ & + T(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta_{n+1} \otimes \Delta) \\ & + T(T \otimes T \otimes T) \circ \rho \circ (id \otimes id \otimes id \otimes \Delta \otimes \Delta_{n+1})], \end{aligned}$$

$$\begin{aligned}
\phi_2 = & [\Delta_{n+1}T + \Delta T_{n+1}] - [(T_{n+1} \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) \\
& + (T \otimes T_{n+1} \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) \\
& + (T \otimes T \otimes T_{n+1}) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta) \\
& + (T \otimes T \otimes T) \circ \rho \circ (\Delta_{n+1} \otimes \Delta \otimes \Delta) \\
& + (T \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta_{n+1} \otimes \Delta) \\
& + (T \otimes T \otimes T) \circ \rho \circ (\Delta \otimes \Delta \otimes \Delta_{n+1})], \\
\phi_3 = & [(\Delta_{n+1} \otimes id \otimes id)\Delta + (\Delta \otimes id \otimes id)\Delta_{n+1}] \\
& - [(id \otimes id \otimes \Delta_{n+1})\Delta + (id \otimes id \otimes \Delta)\Delta_{n+1}].
\end{aligned}$$

Proposition 6.1. *Let (A, T, Δ) be a ternary distributive weak bialgebra and (A_t, T_t, Δ_t) , where $A_t = A \otimes \mathbb{K}[[t]]$, $T_t = T + tT_1 + \cdots + t^n T_n + \cdots : A_t \otimes A_t \otimes A_t \rightarrow A_t$ and $\Delta_t = \Delta + t\Delta_1 + \cdots + t^n \Delta_n + \cdots : A_t \rightarrow A_t \otimes A_t \otimes A_t$ where $T_i : A \otimes A \otimes A \rightarrow A$, $\Delta_i : A \rightarrow A \otimes A \otimes A$, $i = 1, 2, \dots$, are sequences of maps.*

Then $\mathcal{D}^2(T_1, \Delta_1) = (\phi_1, \phi_2, \phi_3) = 0$.

In the sequel, we focus on deformations of a ternary distributive set (Q, T) and set $A = \mathbb{K}[Q]$ to be the vector space spanned by the elements of Q . We refer to $(\mathbb{K}[Q], T)$, where T is extended by \mathbb{K} -trilinearity, as a ternary distributive algebra.

Definition 6.2. A one-parameter formal deformation of $(\mathbb{K}[Q], T)$ is a pair $(\mathbb{K}[Q]_t, T_t)$ where $\mathbb{K}[Q]_t$ is a $\mathbb{K}[[t]]$ -algebra given by $\mathbb{K}[Q]_t = \mathbb{K}[Q] \otimes \mathbb{K}[[t]]$ with all ternary structures inherited by extending those on $\mathbb{K}[Q]_t$ with the identity on the $\mathbb{K}[[t]]$ factor (the trivial deformation as the algebra), with a deformations of T given by $T_t = T + tT_1 + \cdots + t^n T_n + \cdots : \mathbb{K}[Q]_t \otimes \mathbb{K}[Q]_t \otimes \mathbb{K}[Q]_t \rightarrow \mathbb{K}[Q]_t$ where $T_i : \mathbb{K}[Q] \otimes \mathbb{K}[Q] \otimes \mathbb{K}[Q] \rightarrow \mathbb{K}[Q]$, $i = 1, 2, \dots$, are linear maps.

The map T_t satisfies the equation

$$T_t \circ (T_t \otimes id \otimes id) = T_t \circ (T_t \otimes T_t \otimes T_t) \circ \rho \circ (id \otimes id \otimes id \otimes D \otimes D).$$

That is for elements $x, y, z, u, v \in Q$, we have

$$\begin{aligned}
& T_t(T_t(x \otimes y \otimes z) \otimes u \otimes v) = \\
(6.4) \quad & T_t(T_t(x \otimes u \otimes v) \otimes T_t(y \otimes u \otimes v) \otimes T_t(z \otimes u \otimes v)).
\end{aligned}$$

We call the equation (6.4) the deformation equation of the ternary operation T .

6.1 Deformation equation

The deformation equation (6.4) may be written by expanding and collecting the coefficients of t^k as

$$\begin{aligned} \sum_{k=0}^{\infty} t^k \sum_{i=0}^k T_i(T_{k-i}(x \otimes y \otimes z) \otimes u \otimes v) = \\ \sum_{k=0}^{\infty} t^k \sum_{m+n+p+q=k} T_m(T_n(x \otimes u \otimes v) \otimes T_p(y \otimes u \otimes v) \otimes T_q(z \otimes u \otimes v)), \end{aligned}$$

where m, n, p and q are non-negative integers. It yields, for $k = 0, 1, 2, \dots$

$$\begin{aligned} \sum_{i=0}^k T_i(T_{k-i}(x \otimes y \otimes z) \otimes u \otimes v) = \\ \sum_{m+n+p+q=k} T_m(T_n(x \otimes u \otimes v) \otimes T_p(y \otimes u \otimes v) \otimes T_q(z \otimes u \otimes v)). \end{aligned}$$

This infinite system gives the necessary and sufficient conditions for T_t to be a distributive ternary operation. The first problem is to give conditions on T_i so that the deformation T_t is distributive.

The first equation ($k = 0$) is the ternary distributivity condition for T_0 . The second equation ($k = 1$) shows that T_1 satisfies $\delta_m^2(T_1) = 0$. More generally, suppose that T_p is the first non-zero coefficient after T in the deformation T_t . This T_p is called the *infinitesimal* of T_t and should satisfy $\delta_m^2(T_p) = 0$. In order to express it as a 2-cocycle with respect to the cohomology defined above, we consider $(\mathbb{K}[Q], T)$ as a weak bialgebra $(\mathbb{K}[Q], T, D)$ where D is the diagonal map defined on the elements of Q as $D(x) = x \otimes x \otimes x$. Therefore the pair $(T_p, 0)$ is a 2-cocycle with respect to the cohomology of $(\mathbb{K}[Q], T, D)$. We call T_p a 2-cocycle of the ternary distributive algebras cohomology with coefficient in itself.

Theorem 6.3. *The map T_p is a 2-cocycle of the ternary distributive algebras cohomology with coefficient in itself.*

Proof. In the equation (6.4), make the following substitution $k = p$ and $T_1 = \dots = T_{p-1} = 0$. \square

6.1.1 Equivalent and trivial deformations

We characterize the equivalent and trivial deformations of ternary distributive algebras.

Definition 6.4. Let (Q, T_0) be a ternary distributive set and $A = (\mathbb{K}[Q], T_0)$ be a corresponding ternary distributive algebra. Let $T_t = \sum_{i \geq 0} T_i t^i$ and $T'_t = \sum_{i \geq 0} T'_i t^i$ be two deformations of $(\mathbb{K}Q, T_0)$, ($T_0 = T'_0$). We say that they are equivalent if there exists a formal isomorphism $\Phi_t : A[[t]] \rightarrow A[[t]]$ which is a $\mathbb{K}[[t]]$ -linear map that may be written in the form

$$\Phi_t = \sum_{i \geq 0} \Phi_i t^i = id + \Phi_1 t + \Phi_2 t^2 + \dots \quad \text{where } \Phi_i \in \text{End}_{\mathbb{K}}(A) \quad \text{and} \quad \Phi_0 = id,$$

such that

$$(6.5) \quad \Phi_t \circ T_t = T'_t \circ \Phi_t.$$

A deformation T_t of T_0 is said to be trivial if and only if T_t is equivalent to T_0 .

The condition (6.5) may be written $\Phi_t(T_t(x \otimes y \otimes z)) = T'_t(\Phi_t(x) \otimes \Phi_t(y) \otimes \Phi_t(z))$, $\forall x, y, z \in A$, which is equivalent to

$$(6.6) \quad \sum_{i \geq 0} \Phi_i \left(\sum_{j \geq 0} T_j(x \otimes y \otimes z) t^j \right) t^i = \sum_{i \geq 0} T'_i \left(\sum_{j \geq 0} \Phi_j(x) t^j \otimes \sum_{k \geq 0} \Phi_k(y) t^k \otimes \sum_{l \geq 0} \Phi_l(z) t^l \right) t^i,$$

or

$$\sum_{i, j \geq 0} \Phi_i(T_j(x \otimes y \otimes z)) t^{i+j} = \sum_{i, j, k, l \geq 0} T'_i(\Phi_j(x) \otimes \Phi_k(y) \otimes \Phi_l(z)) t^{i+j+k+l}.$$

By identification of coefficients, one obtains that the constant coefficients are identical

$$T_0 = T'_0 \quad \text{because} \quad \Phi_0 = id$$

and for coefficients of t one has

$$\begin{aligned} \Phi_0(T_1(x \otimes y \otimes z)) + \Phi_1(T_0(x \otimes y \otimes z)) = \\ T'_1(\Phi_0(x) \otimes \Phi_0(y) \otimes \Phi_0(z)) + T'_0(\Phi_1(x) \otimes \Phi_0(y) \otimes \Phi_0(z)) + \\ T'_0(\Phi_0(x) \otimes \Phi_1(y) \otimes \Phi_0(z)) + T'_0(\Phi_0(x) \otimes \Phi_0(y) \otimes \Phi_1(z)). \end{aligned}$$

It follows

$$T_1(x \otimes y \otimes z) + \Phi_1(T_0(x \otimes y \otimes z)) = T'_1(x \otimes y \otimes z) + T_0(\Phi_1(x) \otimes y \otimes z) \\ + T_0(x \otimes \Phi_1(y) \otimes z) + T_0(x \otimes y \otimes \Phi_1(z)).$$

Consequently,

$$T'_1(x \otimes y \otimes z) = T_1(x \otimes y \otimes z) + \Phi_1(T_0(x \otimes y \otimes z)) - T_0(\Phi_1(x) \otimes y \otimes z) \\ - T_0(x \otimes \Phi_1(y) \otimes z) - T_0(x \otimes y \otimes \Phi_1(z)).$$

That is $T'_1 = T_1 - \delta_m^1 \Phi_1$. Therefore T_1 and T'_1 are in the same cohomology class. We consider here the cohomology defined in Section 4, for which we set $\Delta = 0$ and $\psi_2 = 0$. Therefore, under the same assumptions, we obtain the following result:

Theorem 6.5. *Let (Q, T_0) be a ternary distributive set, $(A = \mathbb{K}[Q], T_0)$ be a corresponding ternary distributive algebra and T_t be a one parameter family of deformations of T_0 . Then T_t is equivalent to*

$$(6.7) \quad T_t(x \otimes y \otimes z) = T_0(x \otimes y \otimes z) + T'_p(x \otimes y \otimes z)t^p + T'_{p+1}(x \otimes y \otimes z)t^{p+1} + \dots$$

where $T'_p \in Z^2(A, A)$ and $T'_p \notin B^2(A, A)$.

Moreover, if $H^2(A, A) = \{0\}$ then every deformation of A is trivial. The ternary distributive algebra is said rigid.

Proof. Let T_{p-1} be the first nonzero term in the deformation. The deformation equation implies $\delta_m^2 T_{p-1} = 0$ which means $T_{p-1} \in Z^2(A, A)$. If further $T_{p-1} \in B^2(A, A)$, i.e. $T_{p-1} = \delta_m^1 g$ with $g \in \text{Hom}(A, A)$, then using a formal morphism $\Phi_t = id + t^{p-1}g$ we obtain that the deformation T_t is equivalent to the deformation given for all $x, y, z \in A$ by

$$T'_t(x \otimes y \otimes z) = \\ \Phi_t \circ T_t \circ (\Phi_t^{-1}(x) \otimes \Phi_t^{-1}(y) \otimes \Phi_t^{-1}(z)) = \\ T_0(x \otimes y \otimes z) + T'_p(x \otimes y \otimes z)t^{p+1} + \dots$$

and again $T'_p \in Z^2(T_0, T_0)$. If $T'_p \in B^2(T_0, T_0)$, we proceed in the same way to remove it, and so one. \square

We end the paper with the fact that we do not know yet how to use these ternary quandles to obtain invariant of knots and/or knotted surfaces.

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A. Ternary Distributivity from Coalgebras

In the following proposition we give a classification of ternary distributive linear maps $q : A \otimes A \otimes A \rightarrow A$ that are compatible with comultiplication meaning that $\Delta q = (q \otimes q \otimes q)\rho(\Delta \otimes \Delta \otimes \Delta)$. Here $A = \mathbb{K}[Q]$ is a vector space of dimension two with a basis $Q = \{x, y\}$ and Δ is the linearization of the diagonal map D . Here is an outline of the proof of the following proposition. If we set $q(x \otimes x \otimes x) = ax + by$, then we obtain that $a\Delta(x) + b\Delta(y) = (ax + by) \otimes (ax + by) \otimes (ax + by)$ which implies that $a^3 = a$, $b^3 = b$, $a^2b = 0$ and $ab^2 = 0$. Thus the only possible values of $q(x \otimes x \otimes x)$ are 0 , $\pm x$ or $\pm y$. The same holds for the other generators $x \otimes x \otimes y$ etc. With these notations we have the following

Proposition A.1. A linear map $q : A \otimes A \otimes A \rightarrow A$ satisfies equation (2.1) (that is ternary distributive) and compatible with the comultiplication if and only if q is one of the functions indicated via any column in the table below. The values are determined on the basis elements $x \otimes x \otimes x$, through $y \otimes y \otimes y$ as indicated in the following chart

$q(x, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, x, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, y)$	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, y, x)$	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	0	0	0	$\pm x$	$\pm x$
$q(y, y, y)$	0	$\pm x$	$\pm y$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0
$q(x, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, x, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, x)$	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, x, y)$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	0	0	0	0
$q(y, y, x)$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	0
$q(y, y, y)$	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$
$q(x, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, x, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, y)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, x, y)$	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, y, x)$	0	0	$\pm x$	$\pm x$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	0	0	0	$\pm x$	$\pm x$	$\pm x$
$q(y, y, y)$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$

$q(x, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, x, y)$	0	0	0	0	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, x)$	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\mp x$	$\mp x$	$\mp x$	0	0
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm y$	$\mp x$	0	$\pm x$	$\mp y$	0	$\pm y$	$\mp x$	0	$\pm x$	$\mp x$	0
$q(y, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, y)$	0	0	0	0	0	0	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, y, x)$	0	0	0	$\pm y$	$\pm y$	$\pm y$	0	0	0	$\mp y$	$\mp y$	$\mp y$	0	0
$q(y, y, y)$	$\mp y$	$\pm y$	0	$\mp y$	0	$\pm y$	0	0	0	$\mp y$	0	$\pm y$	$\mp y$	0
$q(x, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(x, x, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, y, x)$	0	$\pm x$	$\pm x$	$\pm x$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	0	0	0	$\pm y$	$\pm y$	$\pm y$
$q(x, y, y)$	$\pm x$	$\mp x$	0	$\pm x$	$\mp y$	0	0	$\pm y$	$\mp y$	0	$\pm y$	$\mp y$	0	0
$q(y, x, x)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$q(y, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	0	$\mp x$	0	0	0	0	0	0	0	$\pm x$
$q(y, y, x)$	0	$\pm y$	$\pm y$	$\pm y$	0	$\pm x$	0	0	0	0	0	0	0	$\pm x$
$q(y, y, y)$	$\pm y$	$\mp y$	0	$\pm y$	0	0	0	0	0	0	0	0	0	0
$q(x, x, x)$	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$
$q(x, y, x)$	$\pm y$	$\mp y$	$\mp y$	$\pm y$	$\pm y$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$
$q(x, y, y)$	$\pm y$	$\mp x$	$\pm x$	$\mp x$	$\pm x$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp x$
$q(y, x, x)$	0	$\mp y$	$\mp y$	$\mp y$	$\pm y$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, x, y)$	0	$\mp x$	$\mp x$	$\pm x$	$\pm x$	$\mp y$	$\mp y$	$\pm x$	$\pm x$	$\mp y$	$\mp y$	$\pm x$	$\pm x$	$\mp y$
$q(y, y, x)$	0	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp y$	$\pm x$	$\mp y$	$\pm x$	$\mp y$	$\pm x$	$\mp y$	$\pm x$	$\mp y$
$q(y, y, y)$	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$
$q(x, y, x)$	$\mp x$	$\mp x$	$\mp x$	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm x$	$\mp x$	0	$\pm x$	0	$\pm x$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp x$
$q(y, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp x$	$\mp x$	$\pm y$	$\pm y$	$\mp x$
$q(y, x, y)$	$\pm x$	$\pm x$	$\pm x$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\pm x$	$\mp y$	$\mp x$
$q(y, y, x)$	$\mp y$	$\pm x$	$\pm x$	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$
$q(y, y, y)$	$\pm y$	$\mp x$	$\pm y$	$\mp y$	0	$\pm y$	$\mp y$	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\mp x$	$\mp x$	$\mp x$	0	0	0	0	0	0	0	0	0	0	0
$q(x, y, x)$	$\pm y$	$\pm y$	$\pm y$	$\mp x$	$\mp x$	$\mp x$	0	0	0	0	0	0	0	0
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	$\mp x$	$\mp x$	$\mp x$	0	0	0	0
$q(y, x, x)$	$\mp x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\pm y$	$\mp y$	0	0	0	0
$q(y, x, y)$	$\pm x$	$\mp y$	$\pm x$	0	0	0	0	0	0	0	0	0	0	0
$q(y, y, x)$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	0	0	0	0	0	0	0	0
$q(y, y, y)$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	0	$\pm y$	$\mp y$	$\pm y$	$\mp y$	$\pm y$	0	$\mp y$	0	$\pm y$

$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	0	0	0	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, x)$	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\mp x$	$\mp x$	$\mp x$	0	0	0
$q(x, y, y)$	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$	$\mp x$	0	$\pm x$
$q(y, x, x)$	$\pm y$	$\mp y$	$\mp y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, x, y)$	0	0	0	0	0	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, y, x)$	0	0	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	0	0	0
$q(y, y, y)$	0	$\mp y$	$\pm y$	$\mp y$	$\pm y$	$\mp y$	0	$\pm y$	$\mp y$	0	$\pm y$	$\mp y$	0	$\pm y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, y)$	$\mp x$	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, x, x)$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, x, y)$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(y, y, x)$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$
$q(y, y, y)$	$\mp y$	0	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$
$q(y, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$
$q(y, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm x$	$\pm y$
$q(y, y, x)$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm y$	$\pm y$
$q(y, y, y)$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, y, x)$	$\mp x$	$\mp x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp x$
$q(y, x, x)$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\mp x$	$\mp x$	$\pm y$	$\mp y$
$q(y, x, y)$	$\mp y$	$\mp y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\pm x$	$\mp y$	$\pm x$
$q(y, y, x)$	$\mp y$	$\pm x$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\mp y$	$\pm x$	$\mp y$	$\pm x$
$q(y, y, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, x)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$
$q(x, y, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$
$q(x, y, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp x$	0	$\pm y$	$\pm y$	$\pm y$
$q(y, x, x)$	$\mp x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm x$	$\pm x$	$\pm y$	0	$\mp y$	0	$\mp y$	0	$\pm x$
$q(y, x, y)$	$\mp y$	$\pm y$	$\mp y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm y$	0	$\pm y$	0	$\pm y$	0	$\mp x$
$q(y, y, x)$	$\mp y$	$\pm y$	$\mp y$	$\pm x$	$\pm y$	$\pm x$	$\pm y$	$\pm y$	0	$\pm y$	0	$\pm y$	0	$\mp x$
$q(y, y, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	0	$\mp y$	0	$\mp y$	0	$\pm x$

$q(x, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, y)$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	$\mp x$	$\mp x$	0	0	0
$q(x, y, x)$	0	0	0	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp x$	$\pm x$	$\mp y$	$\mp y$	$\mp y$
$q(x, y, y)$	$\mp y$	0	$\pm y$	$\mp x$	$\pm y$	$\mp y$	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\mp y$	0	$\pm y$
$q(y, x, x)$	0	0	0	$\mp y$	$\mp y$	0	0	$\mp x$	0	$\pm x$	$\mp x$	0	0	0
$q(y, x, y)$	0	0	0	$\pm y$	$\pm y$	0	0	$\pm x$	0	$\mp y$	$\mp y$	0	0	0
$q(y, y, x)$	0	0	0	$\pm y$	$\pm y$	0	0	$\mp x$	0	$\mp y$	$\pm y$	0	0	0
$q(y, y, y)$	0	0	0	$\mp y$	$\mp y$	0	0	$\mp x$	0	$\pm x$	$\mp x$	0	0	0
$q(x, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, y)$	0	0	0	0	0	0	0	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, x)$	0	0	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp x$	$\pm x$	$\pm x$	$\pm x$
$q(x, y, y)$	$\mp y$	0	$\pm y$	$\pm y$	$\pm y$	$\mp y$	0	$\pm y$	$\mp x$	$\pm y$	$\pm y$	$\mp x$	$\pm x$	$\pm y$
$q(y, x, x)$	0	0	$\mp x$	0	$\pm x$	0	0	0	$\mp y$	$\mp y$	$\mp x$	$\mp y$	$\pm y$	$\mp y$
$q(y, x, y)$	0	0	0	0	0	0	0	0	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(y, y, x)$	0	0	0	0	0	0	0	0	$\pm y$	$\pm y$	$\mp y$	$\pm y$	$\pm y$	$\pm y$
$q(y, y, y)$	0	0	$\mp x$	0	$\pm x$	0	0	0	$\mp y$	$\mp y$	$\mp x$	$\mp y$	$\pm y$	$\mp y$
$q(x, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, x, y)$	$\pm x$	$\pm x$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$
$q(x, y, x)$	$\pm x$	$\pm x$	$\pm y$	$\pm y$	$\mp y$	$\mp y$	$\mp y$	$\mp y$	0	0	0	$\pm x$	$\pm x$	$\pm y$
$q(x, y, y)$	$\pm y$	$\pm y$	$\pm x$	$\pm y$	$\mp y$	0	$\pm y$	$\pm y$	$\mp y$	0	$\pm y$	$\pm x$	$\pm y$	$\mp y$
$q(y, x, x)$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	0	0	$\mp x$	0	0	0	0	$\pm y$	$\pm y$	0
$q(y, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	0	0	$\mp x$	0	0	0	0	$\pm y$	$\pm y$	0
$q(y, y, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	0	0	$\mp x$	0	0	0	0	$\pm y$	$\pm y$	0
$q(y, y, y)$	$\pm x$	$\pm y$	$\pm y$	$\pm y$	0	0	$\mp x$	0	0	0	0	$\pm y$	$\pm y$	0
$q(x, x, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$									
$q(x, x, y)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$									
$q(x, y, x)$	$\pm y$	$\pm y$	$\pm y$	$\pm y$	$\pm y$									
$q(x, y, y)$	0	$\pm x$	$\pm y$	$\pm y$	$\pm y$									
$q(y, x, x)$	0	$\pm y$	0	$\pm x$	$\pm y$									
$q(y, x, y)$	0	$\pm y$	0	$\pm x$	$\pm y$									
$q(y, y, x)$	0	$\pm y$	0	$\pm x$	$\pm y$									
$q(y, y, y)$	0	$\pm y$	0	$\pm x$	$\pm y$									

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