Reconfiguring $k$-colourings of Complete Bipartite Graphs

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Abstract. Let $H$ be a graph, and $k \geq \chi(H)$ an integer. We say that $H$ has a cyclic Gray code of $k$-colourings if and only if it is possible to list all its $k$-colourings in such a way that consecutive colourings, including the last and the first, agree on all vertices of $H$ except one. The Gray code number of $H$ is the least integer $k_0(H)$ such that $H$ has a cyclic Gray code of its $k$-colourings for all $k \geq k_0(H)$. For complete bipartite graphs, we prove that $k_0(K_{\ell,r}) = 3$ when both $\ell$ and $r$ are odd, and $k_0(K_{\ell,r}) = 4$ otherwise.

1. Introduction

Let $H$ be a graph and $k$ a positive integer. The $k$-colouring graph of $H$, $G_k(H)$, has as its vertices the proper $k$-colourings of $H$, any two of which are joined by an edge if and only if they agree on all but one vertex of $H$. When this graph is connected, any given $k$-colouring can be reconfigured into any other via a sequence of recolourings which each change the colour of exactly one vertex. When it is hamiltonian, there is a cyclic list that contains all of the $k$-colourings of $H$ and consecutive elements of the list differ in the colour of exactly one vertex.
The Gray code number of $H$, denoted $k_0(H)$, is defined to be the smallest integer $k$ such that $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$; that is, $k_0(H)$ is the least integer such that there exists a cyclic Gray code of $k$-colourings of $H$. It is shown in [7] that for any simple graph $H$, $k_0(H)$ is well-defined; i.e., for $k \geq \text{col}(G) + 2$, where $\text{col}(G)$ denotes the colouring number of $G$, it is always possible to enumerate all proper $k$-colourings of $H$ in such a way that any two successive colourings, including the first and the last, differ on only one vertex. A discussion of the origins of the Gray code number can be found in [7].

For our purposes, a proper $k$-colouring of a graph $H$ is a function $f : V(H) \rightarrow \{1, 2, \ldots, k\}$ such that if $xy \in E(H)$, $f(x) \neq f(y)$. We refer to the function values as the colours of the vertices, and for convenience use the term $k$-colouring (since we only consider proper $k$-colourings). This terminology is consistent with Bondy and Murty [2], and we refer the reader to that text for notation and terminology not defined here.

Choo and MacGillivray [7] establish Gray code numbers for various classes of graphs. For complete graphs, $k_0(K_1) = 3$ and $k_0(K_n) = n + 1$ when $n \geq 2$. For cycles, $k_0(C_n) = 4$ for $n \geq 3$. Any tree $T$ satisfies $k_0(T) = 3$, except if $T$ is a star with an odd number (at least three) of vertices, in which case $k_0(T) = 4$.

The results here extend the work presented in [7] in that we determine the Gray code numbers of complete bipartite graphs, of which stars are a special case. The general case of bipartite graphs that are not complete remains largely unexplored. Connectivity and hamiltonicity of the $k$-colouring graphs of complete multipartite graphs is addressed in [1].

Connectivity of $k$-colouring graphs arises in random sampling of $k$-colourings, and approximating the number of $k$-colourings (see [8, 12, 13]). Neither the 2-colouring graph of a bipartite graph nor the 3-colouring graph of a 3-chromatic graph is ever connected, but for each $k \geq 4$ there exist $k$-chromatic graphs for which the $k$-colouring graph is connected, and others for which it is disconnected [4, 5]. On the other hand, for any graph $H$, the $k$-colouring graph is connected for all $k \geq \text{col}(H) + 1$ [8]. While it is Polynomial to decide if the 3-colouring graph of a bipartite graph is connected [3], it is NP-complete to decide if two given colourings belong to the same component of such a graph [6]. In [3] it is shown that the diameter of any component of the 3-colouring graph of a bipartite graph is bounded by a quadratic function of the number of vertices, but for each $k \geq 4$ there exist bipartite graphs on $n$ vertices for which the diameter of some component of the $k$-colouring graph is exponential in $n$; for each $k \geq 4$ it is PSPACE complete to decide if two given $k$-colourings belong to the same component of the $k$-colouring graph.

Other $k$-colouring graphs have also been considered. Viewing a $k$-colouring of $H$ as a partition of $V(H)$ with at most $k$ cells leads to the $k$-Bell colour graph, while viewing it as a partition into exactly $k$ parts leads to the $k$-Stirling colour graph. Every graph on $n$ vertices has a hamiltonian $n$-Bell colour graph, and for each $k \geq 4$, the $k$-Stirling colour graph of a tree is hamiltonian [9]. The canonical $k$-colouring graph of $H$ with respect to a fixed ordering $\Pi$ of $V(H)$ is the subgraph
of $G_k(H)$ obtained by first defining two $k$-colourings to be equivalent if they give rise to the same partition of $V(H)$, and then taking the subgraph induced by the set of equivalence class representatives which are lexicographically least with respect to $\Pi$. For every tree $T$ there exists an ordering $\Pi$ of the vertices such that the canonical $k$-colouring graph of $T$ with respect to $\Pi$ is Hamiltonian for all $k \geq 3$ [10]. For any graph $H$ and any vertex ordering $\Pi$, the canonical $k$-colouring graph of $H$ with respect to $\Pi$ is a spanning subgraph of the $k$-Bell colour graph of $H$. Finally, connectivity of the graph of list-$L(2,1)$-labellings – proper colourings with some additional restrictions – has recently been studied in [11].

2. Gray Code Numbers of Complete Bipartite Graphs

Let $K_{\ell,r}$ be a complete bipartite graph with bipartition $(L,R)$, where the sets $L$ and $R$ are $L = \{p_1, p_2, \ldots, p_\ell\}$ and $R = \{q_1, q_2, \ldots, q_r\}$, respectively. A colouring of $K_{\ell,r}$ with $f(p_i) = a_i$, $1 \leq i \leq \ell$ and $f(q_i) = b_i$, $1 \leq i \leq r$ is denoted $(a_1 a_2 \ldots a_\ell | b_1 b_2 \ldots b_r)$.

We begin by establishing a lower bound on $k_0(K_{\ell,r})$.

**Theorem 2.1.** For positive integers $\ell$ and $r$, $G_2(K_{\ell,r})$ is not hamiltonian, and $G_3(K_{\ell,r})$ is hamiltonian if and only if $\ell, r$ are both odd.

**Proof.** A 2-colouring of $K_{\ell,r}$ is completely determined by the colour of any one of its vertices, implying that $|V(G_2(K_{\ell,r}))| = 2$. Moreover, these two 2-colourings cannot be joined by an edge since the colours of all vertices of $K_{\ell,r}$ must be changed to obtain one 2-colouring from the other. Since $K_{\ell,r}$ has a least two vertices, $G_2(K_{\ell,r})$ is not connected and hence not hamiltonian.

Notice that every 3-colouring of $K_{\ell,r}$ leaves at least one of $L, R$ monochromatic, so for each $j$, $1 \leq j \leq 3$, we define $L_j$ to be the subgraph of $G_3(H)$ induced by 3-colourings $f$ in which $f(p) = j$ for all $p \in L$; $R_j$ is defined analogously. Thus every vertex of $G_3(H)$ belongs to (at least) one of $L_1, L_2, L_3, R_1, R_2, R_3$.

The colourings in $L_1$ have all vertices of $L$ coloured with 1 and the vertices of $R$ coloured with 2 and 3. Thus each colouring in $L_1$ can be thought of as a binary string of length $r$ over $\{2, 3\}$, implying that $L_1$ is isomorphic to the $r$-dimensional cube, $Q_r$.

It is routine to prove (and also follows from a result in [14]) that $Q_r$ has a Hamilton path between [00...0] and [11...1] if and only if $r$ is odd. Thus if $r$ is odd, there is a Hamilton path $P_{L,1}$ in $L_1$ between (11...1|22...2) and (11...1|33...3). If $\ell$ is also odd, then $R_3 \cong Q_\ell$, so $R_3$ has a Hamilton path $P_{R,3}$ between (11...1|33...3) and (22...2|33...3). Analogously,

- $L_2$ has a Hamilton path $P_{L,2}$ between (22...2|33...3) and (22...2|11...1);
- $R_1$ has a Hamilton path $P_{R,1}$ between (22...2|11...1) and (33...3|11...1);
- $L_3$ has a Hamilton path $P_{L,3}$ between (33...3|11...1) and (33...3|22...2);
- $R_2$ has a Hamilton path $P_{R,2}$ between (33...3|22...2) and (11...1|22...2).
It follows that
\[ P_{L,1} \cup P_{R,3} \cup P_{L,2} \cup P_{R,1} \cup P_{L,3} \cup P_{R,2} \]

is a Hamilton cycle of \( G_3(K_{\ell,r}) \).

Conversely, if \( r \) is even, then \( G_3(K_{\ell,r}) \) is not hamiltonian. The two-vertex set \( \{(11\ldots122\ldots2), (11\ldots133\ldots3)\} \) forms a cut of \( G_3(K_{\ell,r}) \), since one must encounter at least one of these two vertices before leaving or entering \( L_1 \). Therefore, a Hamilton cycle of \( G_3(K_{\ell,r}) \) must contain a Hamilton path of \( L_1 \) that starts and ends at these two vertices. Since \( r \) is even, \( L_1 \cong Q_r \) contains no such Hamilton path, and thus \( G_3(K_{\ell,r}) \) is not hamiltonian. \( \square \)

Theorem 2.1 implies that if \( \ell, r \geq 1 \) and at least one of these is even, then \( k_0(K_{\ell,r}) \geq 4 \). It remains to show that this inequality is an equality.

Consider the complete graph \( K_n \) with vertex set \( \{1, 2, \ldots, n\} \), and the cartesian product \( K_n \square K_n \) with vertex set \( \{(i, j) \mid 1 \leq i, j \leq n\} \). Denote by \( J_n \) the graph obtained from \( K_n \square K_n \) by deleting the set of vertices \( \{(i, i) \mid 1 \leq i \leq n - 1\} \).

Figure 1: Hamilton paths in the graph \( J_n \) of Lemma 2.2 when \( n = 7 \) and \( n = 8 \). Not all edges are shown.
Lemma 2.2. For \( n \geq 3 \), \( J_n \) has a Hamilton path between \( (n, n) \) and any vertex of \( J_n - (n, n) \).

Proof. Let \( v = (n, n) \). In Figure 1, we depict Hamilton paths between \( v \) and \((1, 2)\) when \( n \) is odd and when \( n \) is even, and Hamilton paths between \( v \) and \((1, n)\) when \( n \) is even and when \( n \) is odd. The lemma is proved by showing that for every \( w \in V(J_n) \), \( w \neq v \), there is an automorphism of \( J_n \) that fixes \( v \) and maps \( w \) to either \((1, 2)\) or \((1, n)\).

For any \( \pi \in S_n \), define \( \phi_\pi : V(J_n) \to V(J_n) \) by

\[
\phi_\pi(a, b) = (\pi(a), \pi(b)).
\]

If \( \pi(n) = n \), then it is straightforward to see that \( \phi_\pi \) is an automorphism of \( J_n \).

Suppose \( w = (w_1, w_2) \in V(J_n) \) is such that neither \( w_1 \) nor \( w_2 \) is equal to \( n \). Choose \( \pi = (1, w_1)(2, w_2) \), so that \( \phi_\pi \) is an automorphism of \( J_n \). Then

\[
\phi_\pi(w) = (\pi(w_1), \pi(w_2)) = (1, 2),
\]

and hence \( J_n \) has a Hamilton path between \( v \) and \( w \). If \( w = (w_1, n) \), then choosing \( \pi = (1, w_1) \) again ensures that \( \phi_\pi \) is an automorphism of \( J_n \), and

\[
\phi_\pi(w) = (\pi(w_1), \pi(n)) = (1, n);
\]

i.e., \( J_n \) has a Hamilton path between \( v \) and \( w \). Finally, suppose \( w = (n, w_2) \), and let \( \tau : V(J_n) \to V(J_n) \) be the automorphism of \( J_n \) in which

\[
\tau(a, b) = (b, a).
\]

Choosing \( \pi = (1, w_2) \) ensures that \( \phi_\pi \circ \tau \) is an automorphism of \( J_n \) in which

\[
\phi_\pi \circ \tau(n, w_2) = \phi_\pi(w_2, n) = (\pi(w_2), \pi(n)) = (1, n).
\]

Again, there is a Hamilton path in \( J_n \) between \( v \) and \( w \). \( \square \)

We now use Lemma 2.2 to prove our main theorem.

Theorem 2.3. Let \( 1 \leq \ell \leq r \) and let \( k \geq 4 \). Then \( G_k(K_{\ell, r}) \) is hamiltonian.

Proof. The proof is by induction on \( \ell \). When \( \ell = 1 \), the graph \( K_{1, r} \) is a star, and it is known [7, Corollary 5.6] that \( G_k(K_{1, r}) \) is hamiltonian for \( k \geq 4 \).

For \( \ell \geq 2 \), let \( K_{\ell, r} \) have bipartition \((L, R)\) with \( u \in L \) and \( v \in R \), and let \( H \) denote the graph obtained from \( K_{\ell, r} \) by deleting \( u \) and \( v \). Then \( H \cong K_{\ell-1, r-1} \), and has bipartition \((L', R')\) where \( L' = L \setminus \{u\} \) and \( R' = R \setminus \{v\} \). Suppose \( f_0, f_1, \ldots, f_{N-1}, f_0 \) is a Hamilton cycle in \( G_k(H) \). For \( 0 \leq i \leq N - 1 \), define \( F_i \) to be the subgraph of \( G_k(K_{\ell, r}) \) induced by the colourings that agree with \( f_i \) on \( H \). In what follows, the subscripts of \( f_i \) and \( F_i \) are taken modulo \( N \). Let \( [F_i, F_{i+1}] \) denote the set of edges that have one end in \( F_i \) and the other end in \( F_{i+1} \).
Suppose \( i \in \{0, 1, \ldots, N - 1\} \). A colouring \( t_i \in V(F_i) \) is called a \textit{sink} if it is incident to an edge in \([F_i, F_{i+1}]\). If \( t_i \) is a sink, then it is adjacent to exactly one colouring in \( V(F_{i+1}) \).

**Claim.** For any \( s_i \in V(F_i) \), there exists a sink \( t_i \neq s_i \), and a Hamilton path in \( F_i \) between \( s_i \) and \( t_i \).

**Proof.** Assume that the set of all colours is \( C := \{1, 2, \ldots, k\} \). Let \( U_i(i) \) and \( U_r(i) \) be the sets of colours used in \( L' \) and \( R' \), respectively, under the colouring \( f_i \). Then \( A_t(i) := C \setminus U_r(i) \) and \( A_r(i) := C \setminus U_t(i) \) are the sets of colours available for \( u \) and \( v \), respectively, to extend \( f_i \) to a colouring in \( F_i \).

Since only one vertex of \( H \) changes colour between \( f_i \) and \( f_{i+1} \), at least one of the equalities \( U_t(i+1) = U_t(i) \) or \( U_r(i+1) = U_r(i) \) holds, implying that \( A_t(i+1) = A_t(i) \) or \( A_r(i+1) = A_r(i) \), respectively. Without loss of generality, assume that \( A_r(i+1) = A_r(i) \).

Define \( \alpha_i = |A_t(i)| \), \( \beta_i = |A_r(i)| \), and let \( A_t(i) = \{x_1, x_2, \ldots, x_{\alpha_i}\} \) and \( A_r(i) = \{y_1, y_2, \ldots, y_{\beta_i}\} \). If \( A_t(i+1) \nsubseteq A_t(i) \), then the colour change from \( f_i \) to \( f_{i+1} \) introduces a new colour to \( R' \), i.e., there exists a colour \( x_j \in U_r(i+1) \setminus U_r(i) \). Since only one vertex of \( H \) changes colour between \( f_i \) and \( f_{i+1} \), \( x_j \) is unique and we may assume, without loss of generality, that \( A_t(i) \setminus A_t(i+1) = \{x_1\} \), and hence \( x_1 \in U_r(i+1) \setminus U_r(i) \). It follows that if a colouring \( t_i \in V(F_i) \) is not a sink, then \( t_i(u) = x_1 \).

Let \( d_i := |A_t(i) \cap A_r(i)| \) be the number of colours available to both \( u \) and \( v \) when extending \( f_i \) to a colouring in \( F_i \). Then \( d_i < \min\{\alpha_i, \beta_i\} \) since \( A_t(i), A_r(i) \) each contains colours not found in the other, namely, the colours used in \( U_t(i), U_r(i) \), respectively. Assume \( x_j = y_j \) for all \( j, 1 \leq j \leq d_i \).

If \( d_i = 0 \), then all colours of \( C \) are used in \( f_i \) and \( \{U_t(i), U_r(i)\} \) is a partition of \( C \). It follows that \( U_t(i+1) \subseteq U_r(i) \), and hence \( A_t(i) \subseteq A_t(i+1) \). Since \( A_t(i) = A_t(i+1) \), every colouring in \( V(F_i) \) is a sink. In this case, \( F_i \cong K_{\alpha_i} \sqcup K_{\beta_i} \); since \( \alpha_i + \beta_i \geq 4 \), \( F_i \) is hamiltonian. We obtain a Hamilton path with \( s_i \in V(F_i) \) as one end by deleting an edge incident to \( s_i \) in an arbitrary Hamilton cycle of \( F_i \).

Now suppose \( d_i \geq 1 \); then \( \alpha_i \geq 2 \) and \( \beta_i \geq 2 \). Let \( s_i \in V(F_i) \). In what follows, we construct a Hamilton cycle in \( F_i \) so that on the Hamilton cycle, \( s_i \) is adjacent to a sink \( t_i \). The subsequent deletion of the edge \( s_it_i \) results in the required Hamilton path.

First consider the case when \( \alpha_i = 2 \). Then \( d_i = 1 \), \( x_1 = y_1 \), and \( \beta_i \geq 3 \) (since \( k \geq 4 \) and \( A_t(i) \cup A_r(i) = \{1, 2, \ldots, k\} \)). If \( s_i(v) \neq y_1 \), then we may assume without loss of generality that \( y_2 = s_i(v) \). Figure 2 shows a Hamilton cycle in \( F_i \) when \( \alpha_i = 2 \) and \( \beta_i = 7 \), where the hollow vertices represent sinks. This Hamilton cycle generalizes to arbitrary \( \beta_i \geq 3 \). Notice that if \( s_i(v) = y_1 \) (recall that \( y_1 = x_1 \)), then \( s_i(u) = x_2 \); otherwise, \( s_i(v) = y_2 \). In either case, \( s_i \) is adjacent to a hollow vertex (sink) \( t_i \) on the Hamilton cycle.

Now suppose \( \alpha_i \geq 3 \). Figures 3 and 4 show Hamilton cycles in \( F_i \) when \( \alpha_i = 4 \) and \( \beta_i = 7, 6 \), respectively; again, the hollow vertices are sinks, and the Hamilton
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Colour of $u$

Colour of $v$

Figure 2: $d_i = 1$ and $\alpha_i = 2$.

Colour of $u$

Colour of $v$

Figure 3: $d_i = 3$, $\alpha_i = 4$, and $\beta_i = 7$.

Colour of $u$

Colour of $v$

Figure 4: $d_i = 3$, $\alpha_i = 4$, and $\beta_i = 6$. 
cycles generalize to arbitrary $\alpha_i$ and $\beta_i$ odd/even, respectively. Notice that any $s_i \in V(F_i)$ is adjacent to a hollow vertex (sink) $t_i$ on the Hamilton cycle.

We now describe a Hamilton cycle of $G_k(K_{\ell,r})$ for $r \geq \ell \geq 2$. Choose $f_0 = (1 \ldots 1|22 \ldots 2)$; since $r \geq \ell \geq 2$, $1 \not\in U_r(1)$ and $2 \not\in U_\ell(1)$. Thus $(1 \ldots 1|22 \ldots 2)$ is a sink in $V(F_0)$, so we define $t_0 = (1 \ldots 1|22 \ldots 2)$.

For $1 \leq i \leq N-2$, define $s_i \in V(F_i)$ to be the vertex adjacent to $t_{i-1}$. By our earlier claim, there is a Hamilton path in $F_i$ between $s_i$ and a sink $t_i$. Suppose $s_{N-1}$ is the colouring in $F_{N-1}$ adjacent to $t_{N-2}$. Observe that all vertices of $F_{N-1}$ are sinks since the colours used in $f_0$ are used in $f_{N-1}$. Thus the Hamilton cycle in $F_{N-1}$ (whose existence is guaranteed in the proof of the claim) offers two choices for $t_{N-1}$: the two colourings adjacent to $s_{N-1}$ in the Hamilton cycle. Choose $t_{N-1}$ so that it is not adjacent to $t_0$, and let $s_0$ be the colouring in $F_0$ adjacent to $t_{N-1}$. This choice guarantees that $s_0 \neq t_0$. Since $F_0$ is isomorphic to the graph $J_n$ in Lemma 2.2 with $n = k - 1$, it follows from that lemma that $F_0$ contains a Hamilton path between $s_0$ and $t_0$. The union of the Hamilton paths contained in the union of the $F_i$, $0 \leq i \leq n-1$, along with the edges $t_is_{i+1}$, $0 \leq i \leq n-1$, yields the required Hamilton cycle.

References


