PRIMITIVE IDEALS AND PURE INFINITENESS OF ULTRAGRAPH $C^*$-ALGEBRAS

Hossein Larki

Abstract. Let $\mathcal{G}$ be an ultragraph and let $C^*(\mathcal{G})$ be the associated $C^*$-algebra introduced by Tomforde. For any gauge invariant ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$, we approach the quotient $C^*$-algebra $C^*(\mathcal{G})/I_{(H,B)}$ by the $C^*$-algebra of finite graphs and prove versions of gauge invariant and Cuntz-Krieger uniqueness theorems for it. We then describe primitive gauge invariant ideals and determine purely infinite ultragraph $C^*$-algebras (in the sense of Kirchberg-Rørdam) via Fell bundles.

1. Introduction

In order to bring graph $C^*$-algebras [7] and Exel-Laca algebras [6] together under one theory, Tomforde introduced in [16] the notion of ultragraphs and associated $C^*$-algebras. An ultragraph is basically a directed graph in which the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex. However, the class of ultragraph $C^*$-algebras are strictly larger than the graph $C^*$-algebras as well as the Exel-Laca algebras (see [17, Section 5]). Due to some similarities, some of fundamental results for graph $C^*$-algebras, such as the Cuntz-Krieger and the gauge invariant uniqueness theorems, simplicity, and $K$-theory computation have been extended to the setting of ultragraphs [16,17]. In particular, by constructing a specific topological quiver $Q(\mathcal{G})$ from an ultragraph $\mathcal{G}$, Katsura et al. described some properties of the ultragraph $C^*$-algebra $C^*(\mathcal{G})$ using those of topological quivers [10]. They showed that every gauge invariant ideal of $C^*(\mathcal{G})$ is of the form $I_{(H,B)}$ corresponding to an admissible pair $(H,B)$ in $\mathcal{G}$.

Recall that for any gauge invariant ideal $I_{(H,B)}$ of a graph $C^*$-algebra $C^*(E)$, there is a (quotient) graph $E/(H,B)$ such that $C^*(E)/I_{(H,B)} \cong C^*(E/(H,B))$ (see [1, 2]). So, the class of graph $C^*$-algebras contains such quotients, and results and properties of graph $C^*$-algebras may be applied for their quotients. For examples, some contexts such as simplicity, $K$-theory, primitivity, and topological stable rank are directly related to the structure of ideals and quotients.
Unlike the $C^*$-algebras of graphs and topological quivers [13], there are no known ways in the literature for describing quotients of an ultragraph $C^*$-algebra by structure of the initial ultragraph. So, many graph $C^*$-algebra’s techniques could not be applied for the ultragraph setting, causing some obstacles in studying these $C^*$-algebras. The initial aim of this article is to analyze the structure of the quotient $C^*$-algebras $C^*(\mathcal{G})/I(\mathcal{H},\mathcal{B})$ for any gauge invariant ideal $I(\mathcal{H},\mathcal{B})$ of $C^*(\mathcal{G})$. For the sake of convenience, we first introduce the notion of quotient ultragraph $\mathcal{G}/(\mathcal{H},\mathcal{B})$ and a relative $C^*$-algebra $C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$ such that $C^*(\mathcal{G})/I(\mathcal{H},\mathcal{B}) \cong C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$ and then prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for $C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$. The uniqueness theorems help us to show when a representation of $C^*(\mathcal{G})/I(\mathcal{H},\mathcal{B})$ is injective. We see that the structure of $C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$ is close to that of graph $C^*$-algebras and we can use them to determine primitive gauge invariant ideals. Moreover, in Section 6, we consider the notion of pure infiniteness for ultragraph $C^*$-algebras in the sense of Kirchberg-Rørdam [11] which is directly related to the structure of quotients. We should note that the initial idea for definition of quotient ultragraphs has been inspired from [9].

The present article is organized as follows. We begin in Section 2 by giving some definitions and preliminaries about the ultragraphs and their $C^*$-algebras which will be used in the next sections. In Section 3, for any admissible pair $(\mathcal{H},\mathcal{B})$ in an ultragraph $\mathcal{G}$, we introduce the quotient ultragraph $\mathcal{G}/(\mathcal{H},\mathcal{B})$ and an associated $C^*$-algebra $C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$. For this, the ultragraph $\mathcal{G}$ is modified by an extended ultragraph $\overline{\mathcal{G}}$ and we define an equivalent relation $\sim$ on $\overline{\mathcal{G}}$. Then $\mathcal{G}/(\mathcal{H},\mathcal{B})$ is the ultragraph $\mathcal{G}$ with the equivalent classes $\{[A] : A \in \mathcal{G}^0\}$. In Section 4, by approaching with graph $C^*$-algebras, the gauge invariant and the Cuntz-Krieger uniqueness theorems will be proved for the quotient ultragraphs $C^*$-algebras. Moreover, we see that $C^*(\mathcal{G}/(\mathcal{H},\mathcal{B}))$ is isometrically isomorphic to the quotient $C^*$-algebra $C^*(\mathcal{G})/I(\mathcal{H},\mathcal{B})$.

In Sections 5 and 6, using quotient ultragraphs, some graph $C^*$-algebra’s techniques will be applied for the ultragraph $C^*$-algebras. In Section 5, we describe primitive gauge invariant ideals of $C^*(\mathcal{G})$, whereas in Section 6, we characterize purely infinite ultragraph $C^*$-algebras (in the sense of [11]) via Fell bundles [5,12].

2. Preliminaries

In this section, we review basic definitions and properties of ultragraph $C^*$-algebras which will be needed through the paper. For more details, we refer the reader to [10] and [16].

**Definition 2.1** ([16]). An ultragraph is a quadruple $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ consisting of a countable vertex set $\mathcal{G}^0$, a countable edge set $\mathcal{G}^1$, the source map $s_{\mathcal{G}} : \mathcal{G}^1 \to \mathcal{G}^0$, and the range map $r_{\mathcal{G}} : \mathcal{G}^1 \to P(\mathcal{G}^0) \setminus \{\emptyset\}$, where $P(\mathcal{G}^0)$ is the collection of all subsets of $\mathcal{G}^0$. If $r_{\mathcal{G}}(e)$ is a singleton vertex for each edge $e \in \mathcal{G}^1$, then $\mathcal{G}$ is an ordinary (directed) graph.
For our convenience, we use the notation $G^0$ in the sense of [10] rather than [16, 17]. For any set $X$, a nonempty subcollection of the power set $\mathcal{P}(X)$ is said to be an algebra if it is closed under the set operations $\cap$, $\cup$, and $\setminus$. If $G$ is an ultragraph, the smallest algebra in $\mathcal{P}(G^0)$ containing $\{\{v\} : v \in G^0\}$ and $\{r_G(e) : e \in G^1\}$ is denoted by $G^0$. We simply denote every singleton set $\{v\}$ by $v$. So, $G^0$ may be considered as a subset of $G^0$.

**Definition 2.2.** For each $n \geq 1$, a path $\alpha$ of length $|\alpha| = n$ in $G$ is a sequence $\alpha = e_1 \ldots e_n$ of edges such that $s(e_{i+1}) \in r(e_i)$ for $1 \leq i \leq n - 1$. If also $s(e_1) \in r(e_n)$, $\alpha$ is called a loop or a closed path. We write $\alpha^0$ for the set $\{s_G(e_i) : 1 \leq i \leq n\}$. The elements of $G^0$ are considered as the paths of length zero. The set of all paths in $G$ is denoted by $G^*$. We may naturally extend the maps $s_G, r_G$ on $G^*$ by defining $s_G(A) = r_G(A) = A$ for $A \in G^0$, and $r_G(\alpha) = r_G(e_n), s_G(\alpha) = s_G(e_1)$ for each path $\alpha = e_1 \cdots e_n$.

**Definition 2.3 ([16]).** Let $G$ be an ultragraph. A Cuntz-Krieger $G$-family is a set of partial isometries $\{s_e : e \in G^1\}$ with mutually orthogonal ranges and a set of projections $\{p_A : A \in G^0\}$ satisfying the following relations:

(UA1) $p_0 = 0$, $p_A p_B = p_{A \cap B}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in G^0$,

(UA2) $s_e^* s_e = p_{r_G(e)}$ for $e \in G^1$,

(UA3) $s_e s_e^* \leq p_{r_G(e)}$ for $e \in G^1$, and

(UA4) $p_v = \sum_{s_G(e) = v} s_e s_e^*$ whenever $0 < |s_G^{-1}(v)| < \infty$.

The $C^*$-algebra $C^*(G)$ of $G$ is the (unique) $C^*$-algebra generated by a universal Cuntz-Krieger $G$-family.

By [16, Remark 2.13], we have

$$C^*(G) = \overline{\text{span}} \{ s_{e_1} p_{A_1} s_{e_2}^* : \alpha, \beta \in G^*, A \in G^0, \text{ and } r_G(\alpha) \cap r_G(\beta) \cap A \neq \emptyset \},$$

where $s_{e_1} := s_{e_1} \cdots s_{e_n}$ if $\alpha = e_1 \cdots e_n$, and $s_{e_1} := p_A$ if $A = A$.

**Remark 2.4.** As noted in [16, Section 3], every graph $C^*$-algebra is an ultragraph $C^*$-algebra. Recall that if $E = (E^0, E^1, r_E, s_E)$ is a directed graph, a collection $\{s_e, p_v : v \in E^0, e \in E^1\}$ containing mutually orthogonal projections $p_v$ and partial isometries $s_e$ is called a Cuntz-Krieger $E$-family if

(GA1) $s_e^* s_e = p_{r_E(e)}$ for all $e \in E^1$,

(GA2) $s_e s_e^* \leq p_{r_E(e)}$ for all $e \in E^1$, and

(GA3) $p_v = \sum_{s_E(e) = v} s_e s_e^*$ for every vertex $v \in E^0$ with $0 < |s_E^{-1}(v)| < \infty$.

We denote by $C^*(E)$ the universal $C^*$-algebra generated by a Cuntz-Krieger $E$-family.

By the universal property, $C^*(G)$ admits the gauge action of the unit circle $\mathbb{T}$. By an ideal, we mean a closed two-sided ideal. Using the properties of quiver $C^*$-algebras [10], the gauge invariant ideals of $C^*(G)$ were characterized in [10, Theorem 6.12] via a one-to-one correspondence with the admissible pairs of $G$ as follows.
Definition 2.5. A subset \( H \subseteq G^0 \) is said to be hereditary if the following properties hold:

(H1) \( s_G(e) \in H \) implies \( r_G(e) \in H \) for all \( e \in G^1 \).
(H2) \( A \cup B \subset H \) for all \( A, B \subset H \).
(H3) If \( A \subset H, B \in G^0, \) and \( B \subset A, \) then \( B \subset H \).

Moreover, a subset \( H \subset G^0 \) is called saturated if for any \( v \in G^0 \) with \( 0 < |s_G^{-1}(v)| < \infty \), then \( \{ r_G(e) : s_G(e) = v \} \subset H \) implies \( v \in H \). The saturated hereditary closure of a subset \( H \subset G^0 \) is the smallest hereditary and saturated subset \( \overline{H} \) of \( G^0 \) containing \( H \).

Let \( H \) be a saturated hereditary subset of \( G^0 \). The set of breaking vertices of \( H \) is denoted by

\[
B_H := \{ w \in G^0 : |s_G^{-1}(w)| = \infty \text{ but } 0 < |r_G(s_G^{-1}(w)) \cap (G^0 \setminus H)| < \infty \}.
\]

An admissible pair \((H, B)\) in \( G \) is a saturated hereditary set \( H \subset G^0 \) together with a subset \( B \subset B_H \). For any admissible pair \((H, B)\) in \( G \), we define the ideal \( I_{(H,B)} \) of \( C^*(G) \) generated by

\[
\{ p_A : A \in G^0 \} \cup \{ p_w^H : w \in B \},
\]

where \( p_w^H := p_w - \sum_{s_G(e)=w, r_G(e)\notin H} s_e s_e^* \). Note that the ideal \( I_{(H,B)} \) is gauge invariant and [10, Theorem 6.12] implies that every gauge invariant ideal \( I \) of \( C^*(G) \) is of the form \( I_{(H,B)} \) by setting

\[
H := \{ A : p_A \in I \} \text{ and } B := \{ w \in B_H : p_w^H \in I \}.
\]

3. Quotient ultragraphs and their \( C^* \)-algebras

In this section, for any admissible pair \((H, B)\) in an ultragraph \( G \), we introduce the quotient ultragraph \( G/(H, B) \) and its relative \( C^* \)-algebra \( C^*(G/(H, B)) \). We will show in Proposition 4.6 that \( C^*(G/(H, B)) \) is isomorphic to the quotient \( C^* \)-algebra \( C^*(\overline{G}/I_{(H,B)}) \).

Let us fix an ultragraph \( G = (G^0, G^1, r_G, s_G) \) and an admissible pair \((H, B)\) in \( G \). For defining our quotient ultragraph \( G/(H, B) \), we first modify \( G \) by an extended ultragraph \( \overline{G} \) such that their \( C^* \)-algebras coincide. For this, add the vertices \( \{ w' : w \in B_H \setminus B \} \) to \( G^0 \) and denote \( \overline{G} := A \cup \{ w' : w \in A \cap (B_H \setminus B) \} \) for each \( A \subset G^0 \). We now define the new ultragraph \( \overline{G} = (\overline{G}^0, \overline{G}^1, r_{\overline{G}}, s_{\overline{G}}) \) by

\[
\overline{G}^0 := G^0 \cup \{ w' : w \in B_H \setminus B \},
\]

\[
\overline{G}^1 := G^1,
\]

the source map

\[
s_{\overline{G}}(e) := \begin{cases} (s_G(e))' & \text{if } s_G(e) \in B_H \setminus B \text{ and } r_G(e) \in H, \\ s_G(e) & \text{otherwise,} \end{cases}
\]

and the rang map \( r_{\overline{G}}(e) := r_G(e) \) for every \( e \in G^1 \). In Proposition 3.3 below, we will see that the \( C^* \)-algebras of \( G \) and \( \overline{G} \) coincide.
**Example 3.1.** Suppose $G$ is the ultragraph

\[
\begin{array}{c}
\node (u) at (0,0) [shape=diamond,draw] {u} ; \\
\node (v) at (2,2) [shape=diamond,draw] {v} ; \\
\node (w) at (2,-2) [shape=diamond,draw] {w} ; \\
\node (w') at (4,-2) [shape=diamond,draw] {w'} ; \\
\node (A) at (4,0) [shape=diamond,draw,fill=white] {A} ; \\
\node (H) at (6,0) [shape=diamond,draw,fill=white, dashed] {H} ; \\
\node (e) at (1,1) [shape=diamond,draw] {e} ; \\
\node (f) at (3,1) [shape=diamond,draw] {f} ; \\
\node (g) at (1,-1) [shape=diamond,draw] {g} ; \\
\node (∞) at (2,0) [shape=diamond,draw] {(∞)} ; \\
\end{array}
\]

where \((∞)\) indicates infinitely many edges. If $H$ is the saturated hereditary subset of $G^0$ containing \(\{v\}\) and $A$, then we have $B_H = \{w\}$. For $B := \emptyset$, consider the admissible pair $(H, \emptyset)$ in $G$. Then the ultragraph $\mathcal{G}$ associated to $(H, \emptyset)$ would be

\[
\begin{array}{c}
\node (u) at (0,0) [shape=diamond,draw] {u} ; \\
\node (v) at (2,2) [shape=diamond,draw] {v} ; \\
\node (w) at (2,-2) [shape=diamond,draw] {w} ; \\
\node (w') at (4,-2) [shape=diamond,draw] {w'} ; \\
\node (A) at (4,0) [shape=diamond,draw,fill=white] {A} ; \\
\node (H) at (6,0) [shape=diamond,draw,fill=white, dashed] {H} ; \\
\node (e) at (1,1) [shape=diamond,draw] {e} ; \\
\node (f) at (3,1) [shape=diamond,draw] {f} ; \\
\node (g) at (1,-1) [shape=diamond,draw] {g} ; \\
\node (∞) at (2,0) [shape=diamond,draw] {(∞)} ; \\
\end{array}
\]

Indeed, since $B_H \setminus B = \{w\}$, for constructing $\mathcal{G}$ we first add a vertex $w'$ to $G$. We then define $\tau_G(f) := A = A$, $\tau_G(e) := \{v, w\} = \{v, w, w'\}$, and $\tau_G(g) := \{u\} = \{u\}$. For the source map $s_G$, for example, since $s_G(f) \in B_H \setminus B$ and $r_G(f) \in H$, we may define $s_G(f) := w'$. Note that the range of each edge emitted by $w'$ belongs to $H$.

As usual, we write $\mathcal{G}^0$ for the algebra generated by the elements of $\mathcal{G}^0 \cup \{\tau_G(e) : e \in \mathcal{G}^1\}$. Note that $A = A$ for every $A \in H$, and hence, $H$ would be a saturated hereditary subset of $\mathcal{G}^0$ as well. Moreover, the set of breaking vertices of $H$ in $\mathcal{G}$ coincides with $B$ (meaning $B^0_H = B$).

**Remark 3.2.** Suppose that $C^*(\mathcal{G})$ is generated by a Cuntz-Krieger $\mathcal{G}$-family $\{s_v, P_A : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$. If a family $M = \{S_v, P_v, P_A : v \in \mathcal{G}^0, A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ in a $C^*$-algebra $X$ satisfies relations (UA1)-(UA4) in Definition 2.3, we may generate a Cuntz-Krieger $\mathcal{G}$-family $N = \{S_v, P_A : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ in $X$. For this, since $\mathcal{G}^0$ is the algebra generated by $\{v, w', \tau_G(e) : v \in \mathcal{G}^0, w \in$
Let \( G \) be an ultragraph, and let \((H,B)\) be an admissible pair in \( G \). If \( G \) is the extended ultragraph as above, then \( C^*(G) \cong C^*(\mathcal{G}) \).

Proof. Suppose that \( C^*(G) = C^*(t_e, q_A) \) and \( C^*(\mathcal{G}) = C^*(s_e, p_C) \). If we define

\[
P_v := q_v \quad \text{for } v \in G^0 \setminus (B_H \setminus B),
\]
\[
P_w := \sum_{r \in \mathcal{G}(e) = w} t_e^* r \quad \text{for } w \in B_H \setminus B,
\]
\[
P_w' := q_w - \sum_{r \in \mathcal{G}(e) = w} t_e^* r \quad \text{for } w \in B_H \setminus B,
\]
\[
P_A := q_A \quad \text{for } A \in \mathcal{G}^0,
\]
\[
S_e := t_e \quad \text{for } e \in \mathcal{G}^1,
\]

then, by Remark 3.2, the family

\[
\{ P_v, P_w, P_w', P_A, S_e : v \in G^0 \setminus (B_H \setminus B), \ w \in B_H \setminus B, \ A \in \mathcal{G}^0, \ e \in \mathcal{G}^1 \}
\]

induces a Cuntz-Krieger \( \mathcal{G} \)-family in \( C^*(G) \). Since all vertex projections of this family are nonzero (which follows all set projections \( P_A \) are nonzero for \( \emptyset \neq A \in \mathcal{G}^0 \)), the gauge-invariant uniqueness theorem [16, Theorem 6.8] implies that the \(*\)-homomorphism \( \phi : C^*(\mathcal{G}) \to C^*(G) \) with \( \phi(p_e) = P_e \) and \( \phi(s_e) = S_e \) is injective. On the other hand, the family generates \( C^*(G) \), and hence, \( \phi \) is an isomorphism. \( \square \)

To define a quotient ultragraph \( \mathcal{G}/(H,B) \), we use the following equivalent relation on \( \mathcal{G} \).

Definition 3.4. Suppose that \((H,B)\) is an admissible pair in \( \mathcal{G} \), and that \( \mathcal{G} \) is the extended ultragraph as above. We define the relation \( \sim \) on \( \mathcal{G} \) by

\[
A \sim C \iff \exists V \in H \text{ such that } A \cup V = C \cup V.
\]

Note that \( A \sim C \) if and only if both sets \( A \setminus C \) and \( C \setminus A \) belong to \( H \).

The following lemma may be proved by a tedious, but straightforward computations.

Lemma 3.5. The relation \( \sim \) is an equivalent relation on \( \mathcal{G} \). Furthermore, the operations

\[
[A] \cup [C] := [A \cup C], \ [A] \cap [C] := [A \cap C], \text{ and } [A] \setminus [C] := [A \setminus C]
\]

for all \( A, C \in \mathcal{G} \).
are well-defined on the equivalent classes \{[A]: A \in \mathcal{G}^0\}.

**Definition 3.6.** Let \(\mathcal{G}\) be an ultragraph, let \((H, B)\) be an admissible pair in \(\mathcal{G}\), and consider the equivalent relation of Definition 3.4 on the extended ultragraph \(\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, \mathcal{G}^2, \mathcal{T}_r, \mathcal{T}_s)\). The **quotient ultragraph of \(\mathcal{G}\) by \((H, B)\)** is the quintuple \(\mathcal{G}/(H, B) = (\Phi(\mathcal{G}^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)\), where

\[
\Phi(\mathcal{G}^0) := \{[v]: v \in \mathcal{G}^0 \setminus H\} \cup \{[w]: w \in B_H \setminus B\},
\]

\[
\Phi(\mathcal{G}^0) := \\{[A]: A \in \mathcal{G}^0\},
\]

\[
\Phi(\mathcal{G}^1) := \{e \in \mathcal{G}^1: \mathcal{T}_r(e) \notin H\},
\]

and \(r: \Phi(\mathcal{G}^1) \to \Phi(\mathcal{G}^0)\), \(s: \Phi(\mathcal{G}^1) \to \Phi(\mathcal{G}^0)\) are the range and source maps defined by

\[
r(e) := \mathcal{T}_r(e) \quad \text{and} \quad s(e) := \mathcal{T}_s(e).
\]

We refer to \(\Phi(\mathcal{G}^0)\) as the vertices of \(\mathcal{G}/(H, B)\).

**Remark 3.7.** Lemma 3.5 implies that \(\Phi(\mathcal{G}^0)\) is the smallest algebra containing

\[
\{[v], [w]: v \in \mathcal{G}^0 \setminus H, w \in B_H \setminus B\} \cup \{[\mathcal{T}_r(e)]: e \in \mathcal{G}^1\}.
\]

**Notation.**

1. For every vertex \(v \in \mathcal{G}^0 \setminus H\), we usually denote \([v]\) instead of \([v]\).
2. For \(A, C \in \mathcal{G}^0\), we write \([A] \subseteq [C]\) whenever \([A] \cap [C] = [A]\).
3. Through the paper, we will denote the range and the source maps of \(\mathcal{G}\) by \(r_{\mathcal{G}}, s_{\mathcal{G}}\), those of \(\mathcal{G}/(H, B)\) by \(r_{\mathcal{G}/(H, B)}, s_{\mathcal{G}/(H, B)}\), and those of \(\mathcal{G}/(H, B)\) by \(r, s\).

Now we introduce representations of quotient ultragraphs and their relative \(C^*-\)algebras.

**Definition 3.8.** Let \(\mathcal{G}/(H, B)\) be a quotient ultragraph. A **representation of \(\mathcal{G}/(H, B)\)** is a set of partial isometries \(\{T_e: e \in \Phi(\mathcal{G}^1)\}\) and a set of projections \(\{Q_{[A]}: [A] \in \Phi(\mathcal{G}^0)\}\) which satisfy the following relations:

(QA1) \(Q_{[\emptyset]} = 0\), and for \([A], [C] \in \Phi(\mathcal{G}^0)\), \(Q_{[A \cap C]} = Q_{[A]} Q_{[C]}\) and \(Q_{[A \cup C]} = Q_{[A]} + Q_{[C]} - Q_{[A \cap C]}\).

(QA2) \(T_e T_f = \delta_{e,f} Q_{r(e)}\) for \(e, f \in \Phi(\mathcal{G}^1)\).

(QA3) \(T_e^* T_e \leq Q_{s(e)}\) for \(e \in \Phi(\mathcal{G}^1)\).

(QA4) \(Q_{[v]} = \sum_{e(s(e) = [v])} T_e T_e^*\), whenever \(0 < |s^{-1}([v])| < \infty\).

We denote by \(C^*(\mathcal{G}/(H, B))\) the universal \(C^*-\)algebra generated by a representation \(\{t_e, q_{[A]}: [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}\) which exists by Theorem 3.10 below.

Note that if \(\alpha = e_1 \cdots e_n\) is a path in \(\mathcal{G}\) such that \(r_{\mathcal{G}}(\alpha) \notin H\), then the hereditary property of \(H\) yields \(\mathcal{T}_r(e_i) \notin H\), and so \(e_i \in \Phi(\mathcal{G}^1)\) for all \(1 \leq i \leq n\). In this case, we denote \(t_\alpha := t_{e_1} \cdots t_{e_n}\). Moreover, we define

\[
(\mathcal{G}/(H, B))^* := \{[A]: [A] \neq [0]\} \cup \left\{\alpha \in \mathcal{G}^*: r(\alpha) \neq [0]\right\}
\]
as the set of finite paths in $\mathcal{G}/(H,B)$ and we can extend the maps $s, r$ on $(\mathcal{G}/(H,B))^*$ by setting

$$s([A]) := r([A]) := [A] \text{ and } s(\alpha) := s(e_1), \ r(\alpha) := r(e_n).$$

The proof of next lemma is similar to the arguments of [16, Lemmas 2.8 and 2.9].

**Lemma 3.9.** Let $\mathcal{G}/(H,B)$ be a quotient ultragraph and let $\{T_e, Q_{[A]}\}$ be a representation of $\mathcal{G}/(H,B)$. Then any nonzero word in $T_e, Q_{[A]}$, and $T_e^*$ may be written as a finite linear combination of the forms $T_{\alpha}Q_{[A]}T_{\beta}$ for $\alpha, \beta \in (\mathcal{G}/(H,B))^*$ and $[A] \in \Phi(\mathcal{G}^0)$ with $[A] \cap r(\alpha) \cap r(\beta) \neq \{0\}$.

**Theorem 3.10.** Let $\mathcal{G}/(H,B)$ be a quotient ultragraph. Then there exists a (unique up to isomorphism) $C^*$-algebra $C^*(\mathcal{G}/(H,B))$ generated by a universal representation $\{t_e, q_{[A]}: [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ for $\mathcal{G}/(H,B)$. Furthermore, all the $t_e$'s and $q_{[A]}$'s are nonzero for $[0] \neq [A] \in \Phi(\mathcal{G}^0)$ and $e \in \Phi(\mathcal{G}^1)$.

**Proof.** By a standard argument similar to the proof of [16, Theorem 2.11], we may construct such universal $C^*$-algebra $C^*(\mathcal{G}/(H,B))$. Note that the universality implies that $C^*(\mathcal{G}/(H,B))$ is unique up to isomorphism. To show the last statement, we generate an appropriate representation for $\mathcal{G}/(H,B)$ as follows. Suppose $C^*(\mathcal{G}) = C^*(s_e, p_A)$ and consider $I_{(H,B)}$ as an ideal of $C^*(\mathcal{G})$ by the isomorphism in Proposition 3.3. If we define

$$
\begin{cases}
Q_{[A]} := p_A + I_{(H,B)} & \text{for } [A] \in \Phi(\mathcal{G}^0), \\
T_e := s_e + I_{(H,B)} & \text{for } e \in \Phi(\mathcal{G}^1),
\end{cases}
$$

then the family $\{T_e, Q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ is a representation for $\mathcal{G}/(H,B)$ in the quotient $C^*\mathcal{G}/I_{(H,B)}$. Note that the definition of $Q_{[A]}$’s is well-defined. Indeed, if $A_1 \cup V = A_2 \cup V$ for some $V \in H$, then $p_{A_1} + q_{V \setminus A_1} = p_{A_2} + q_{V \setminus A_2}$ and hence $p_{A_1} + I_{(H,B)} = p_{A_2} + I_{(H,B)}$ by the facts $V \setminus A_1, V \setminus A_2 \in H$.

Moreover, all elements $Q_{[A]}$ and $T_e$ are nonzero for $[0] \neq [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)$. In fact, if $Q_{[A]} = 0$, then $p_A \in I_{(H,B)}$ and we get $A \in H$ by [10, Theorem 6.12]. Also, since $T_e^2 T_e = Q_{r(e)} \neq 0$, all partial isometries $T_e$ are nonzero.

Now suppose that $C^*(\mathcal{G}/(H,B))$ is generated by the family $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$. By the universality of $C^*(\mathcal{G}/(H,B))$, there is a $\sim$-homomorphism $\phi : C^*(\mathcal{G}/(H,B)) \to C^*(\mathcal{G})/I_{(H,B)}$ such that $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$, and thus, all elements of $\{t_e, q_{[A]} : [0] \neq [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ are nonzero. \hfill $\square$

Note that, by a routine argument, one may obtain

$$C^*(\mathcal{G}/(H,B)) = \overline{\text{span}} \{t_{\alpha} q_{[A]} t_{\beta}^* : \alpha, \beta \in (\mathcal{G}/(H,B))^*, r(\alpha) \cap [A] \cap r(\beta) \neq \{0\}\}.$$
4. Uniqueness theorems

After defining the $C^*$-algebras of quotient ultragraphs, in this section, we prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for them. To do this, we approach to a quotient ultragraph $C^*$-algebra by graph $C^*$-algebras and then apply the corresponding uniqueness theorems for graph $C^*$-algebras. This approach is a developed version of the dual graph method of [14, Section 2] and [16, Section 5] with more complications. In particular, we show that the $C^*$-algebra $C^*(\mathcal{G}/(H, B))$ is isomorphic to the quotient $C^*(\mathcal{G})/I_{(H, B)}$, and the uniqueness theorems may applied for such quotients.

We fix again an ultragraph $\mathcal{G}$, an admissible pair $(H, B)$ in $\mathcal{G}$, and the quotient ultragraph $\mathcal{G}/(H, B) = (\Phi(G^0), \Phi(G^1), r, s)$.

**Definition 4.1.** We say that a vertex $[v] \in \Phi(G^0)$ is a sink if $s^{-1}([v]) = \emptyset$. If $[v]$ only emits finitely many edges of $\Phi(G^1)$, $[v]$ is called a regular vertex. Any non-regular vertex is called a singular vertex. The set of singular vertices in $\Phi(G^0)$ is denoted by

$$\Phi_{\text{sg}}(G^0) := \{ [v] \in \Phi(G^0) : |s^{-1}([v])| = 0 \text{ or } \infty \}.$$  

Let $F$ be a finite subset of $\Phi_{\text{sg}}(G^0) \cup \Phi(G^1)$. Write $F^0 := F \cap \Phi_{\text{sg}}(G^0)$ and $F^1 := F \cap \Phi(G^1) = \{ e_1, \ldots, e_n \}$. We want to construct a special graph $G_F$ such that $C^*(G_F)$ is isomorphic to $C^*(I_e, q_{[v]} : [v] \in F^0, e \in F^1)$. For each $\omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n \setminus \{0^n\}$, we write

$$r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j) \quad \text{and} \quad R(\omega) := r(\omega) \setminus \bigcup_{[v] \in F^0} [v].$$  

Note that $r(\omega) \cap r(\nu) = \emptyset$ for distinct $\omega, \nu \in \{0,1\} \setminus \{0^n\}$. If

$$\Gamma_0 := \{ \omega \in \{0,1\}^n \setminus \{0^n\} : \exists [v_1], \ldots, [v_m] \in \Phi(G^0) \text{ such that}$$

$$R(\omega) = \bigcup_{i=1}^m [v_i] \text{ and } \emptyset \neq s^{-1}([v_i]) \subseteq F^1 \text{ for } 1 \leq i \leq m \},$$

we consider the finite set

$$\Gamma := \{ \omega \in \{0,1\}^n \setminus \{0^n\} : R(\omega) \neq \emptyset \text{ and } \omega \notin \Gamma_0 \}.$$  

Now we define the finite graph $G_F = (G_F^0, G_F^1, r_F, s_F)$ containing the vertices $G_F^0 := F^0 \cup F^1 \cup \Gamma$ and the edges

$$G_F^1 := \{ (e, f) \in F^1 \times F^1 : s(f) \subseteq r(e) \} \cup \{ (e, [v]) \in F^1 \times F^0 : [v] \subseteq r(e) \} \cup \{ (e, \omega) \in F^1 \times \Gamma : \omega_i = 1 \text{ when } e = e_i \}$$

with the source map $s_F(e, f) = s_F(e, [v]) = s_F(e, \omega) = e$, and the range map $r_F(e, f) = f$, $r_F(e, [v]) = [v]$, $r_F(e, \omega) = \omega$.  

We can use these relations to get $\Phi(\mathcal{G})$ a vertex of $\Phi(\mathcal{G})$. Let $\Phi(\mathcal{G}) = \Phi(\mathcal{G})$. If $\Phi(\mathcal{G}) = \Phi(\mathcal{G})$, then the elements

$$Q_e := t_e t_e^*, \quad Q_{[v]} := q_{r_e}(1 - \sum_{e \in E} t_e t_e^*)$$

form a Cuntz-Krieger $\mathcal{G}_F$-family generating the $C^*$-subalgebra $C^*(t_e, q_{[v]} : [v] \in F^0, e \in F^1)$ of $C^*(\mathcal{G}(H, B))$. Moreover, all projections $Q_e$ are nonzero.

**Proof.** We first note that all the projections $Q_e$, $Q_{[v]}$, and $Q_{\omega}$ are nonzero. Indeed, each $[v] \in F^0$ is a singular vertex in $\mathcal{G}(H, B)$, so $Q_{[v]}$ is nonzero. Also, by definition, for every $\omega \in \Gamma$ we have $\omega \not\in \Gamma_0$ and $R(\omega) \neq \emptyset$. Hence, for any $\omega \in \Gamma$, if there is an edge $f \in \Phi(\mathcal{G}) \setminus F^1$ with $s(f) \subseteq R(\omega)$, then $0 \neq t_f t_f^* \leq Q_{\omega}$. If there is a sink $[w]$ such that $[w] \subseteq R(\omega) \setminus \bigcup F^0$, then $0 \neq q_{[w]} \leq q_{R(\omega)}(1 - \sum_{e \in F^1} t_e t_e^*) = Q_{\omega}$. Thus $Q_{\omega}$ is nonzero in either case. In addition, the projections $Q_e$, $Q_{[v]}$, and $Q_{\omega}$ are mutually orthogonal because of the factor $1 - \sum_{e \in F^1} t_e t_e^*$ and the definition of $R(\omega)$.

Now we show the collection $\{T_e, Q_a : a \in G^0, x \in G^1\}$ is a Cuntz-Krieger $\mathcal{G}_F$-family by checking the relations (GA1)-(GA3) in Remark 2.4.

(GA1): Since $Q_{[v]}, Q_{\omega} \leq q_{r_e}(e, [v]), (e, \omega) \in G^1_F$, we have

$$T_{(e,f)}T_{(e,f)} = Q_{(e,f)}t_e t_e^* = t_f t_f^* q_{r_e}(e, f) t_f t_f^* = Q_f,$$

and

$$T_{(e,[v])}T_{(e,[v])} = Q_{[v]}t_e t_e^* Q_{[v]} = Q_{[v]} q_{r_e} Q_{[v]} = Q_{[v]},$$

and

$$T_{(e,[\omega])}T_{(e,[\omega])} = Q_{\omega} t_e t_e^* Q_{\omega} = Q_{\omega} q_{r_e} Q_{\omega} = Q_{\omega}.$$

(GA2): This relation may be checked similarly.

(GA3): Note that any element of $F^0 \cup \Gamma$ is a sink in $\mathcal{G}_F$. So, fix some $e_i \in F^1$ as a vertex of $G^1_F$. Write $q_{F^0} := \sum_{[v] \in F^0} q_{[v]}$. We compute

(i) $q_{r_{(e_i)}} \sum_{f \in F^1} q_{(e_i)} \sum_{s(f) \subseteq r_{(e_i)}} t_f t_f^* = q_{r_{(e_i)}} \sum_{f \in F^1} t_f t_f^*;

(ii) q_{r_{(e_i)}} \sum_{[v] \in F^0, [v] \subseteq r_{(e_i)}} Q_{[v]} = q_{r_{(e_i)}} \sum_{[v] \in F^0} q_{[v]}(1 - \sum_{e \in F^1} t_e t_e^*)

= q_{r_{(e_i)}} q_{F^0}(1 - \sum_{e \in F^1} t_e t_e^*);

(iii) $q_{\omega} = \sum_{\omega \in \Gamma, \omega_i = 1} q_{R(\omega)}(1 - \sum_{e \in F^1} t_e t_e^*) = \sum_{\omega_i = 1} q_{R(\omega)}(1 - \sum_{e \in F^1} t_e t_e^*),$

because $\sum_{\omega_i = 1} q_{R(\omega)} = q_{r_{(e_i)}}(1 - q_{F^0})$.

We can use these relations to get

$$T_{(e,f)} + \sum_{s(f) \subseteq r_{(e_i)}} T_{(e,[v])} + \sum_{\omega \in \Gamma, \omega_i = 1} T_{(e,[\omega])}.$$
Corollary 4.3. If $F$ is a finite subset of $G_F$ (i.e., $|\{x \in G_F^1 : s_F(x) = e_i\}| > 0$), we conclude that
\[
\sum_{f \in F^1, s [f] \subseteq r (e_i)} T_{(e_i,f)} T^*_{(e_i,f)} + \sum_{[v] \in F^0, [v] \subseteq r (e_i)} T_{(e_i,[v])} T^*_{(e_i,[v])}
\]

for every $e_i \in F^1$. Also, for each $[v] \in F^0$, we have
\[
Q_{[v]} + \sum_{e \in F^1, s (e) = [v]} Q_e = t_{[v]} (1 - \sum_{e \in F^1} t_e t^*_e) + \sum_{e \in F^1, s (e) = [v]} t_e t^*_e
\]

which establishes the relation (GA3).

Furthermore, equation (4.1) in above says that $t_{e_i} \in C^*(T_s, Q_s)$ for every $e_i \in F^1$. Also, for each $[v] \in F^0$, we have
\[
Q_{[v]} + \sum_{e \in F^1, s (e) = [v]} Q_e = t_{[v]} (1 - \sum_{e \in F^1} t_e t^*_e) + \sum_{e \in F^1, s (e) = [v]} t_e t^*_e
\]

Therefore, the family $\{T_x, Q_a : a \in G^0_F, x \in G^1_F\}$ generates the $C^*$-subalgebra $C^*(\{t_e, q_{[v]} : e \in F^1, [v] \in F^0\})$ of $C^*(\hat{G}/(H, B))$ and the proof is complete. □

Corollary 4.3. If $F$ is a finite subset of $\Phi_{sg}(G^0) \cup \Phi(G^1)$, then $C^*(G_F)$ is isometrically isomorphic to the $C^*$-subalgebra of $C^*(\hat{G}/(H, B))$ generated by $\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\}$. □

Proof. Suppose that $X$ is the $C^*$-subalgebra generated by $\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\}$ and let $\{T_x, Q_a : a \in G^0_F, x \in G^1_F\}$ be the Cuntz-Krieger $G_F$-family in Proposition 4.2. If $C^*(G_F) = C^*(s_x, p_a)$, then there exists a $*$-homomorphism $\phi : C^*(G_F) \to X$ with $\phi(p_a) = Q_a$ and $\phi(s_x) = T_x$ for every $a \in G^0_F, x \in G^1_F$. Since each $Q_a$ is nonzero by Proposition 4.2, the gauge invariant uniqueness theorem implies that $\phi$ is injective. Moreover, the family $\{T_x, Q_a\}$ generates $X$, so $\phi$ is an isomorphism. □
Note that if $F_1 \subseteq F_2$ are two finite subsets of $\Phi_{sg}(G^0) \cup \Phi(G^1)$ and $X_1, X_2$ are the $C^*$-subalgebras of $C^*(G/(H, B))$ associated to $G_{F_1}$ and $G_{F_2}$, respectively, we then have $X_1 \subseteq X_2$ by Proposition 4.2.

Remark 4.4. Using relations (QA1)-(QA4) in Definition 3.8, each $q_{[A]}$ for $[A] \in \Phi(G^0)$, can be produced by the elements of

$$\{ q_{[v]} : [v] \in \Phi_{sg}(G^0) \} \cup \{ t_e : e \in \Phi(G^1) \}$$

with finitely many operations. So, the $*$-subalgebra of $C^*(G/(H, B))$ generated by

$$\{ q_{[v]} : [v] \in \Phi_{sg}(G^0) \} \cup \{ t_e : e \in \Phi(G^1) \}$$

is dense in $C^*(G/(H, B))$.

As for graph $C^*$-algebras, we can apply the universal property to have a strongly continuous gauge action $\gamma : \mathbb{T} \to \text{Aut}(C^*(G/(H, B)))$ such that

$$\gamma_z(t_e) = z t_e \text{ and } \gamma_z(q_{[A]}) = q_{[A]}$$

for every $[A] \in \Phi(G^0)$, $e \in \Phi(G^1)$, and $z \in \mathbb{T}$. Now we are ready to prove the uniqueness theorems.

**Theorem 4.5** (The Gauge Invariant Uniqueness Theorem). Let $G/(H, B)$ be a quotient ultragraph and let $\{ T_e, Q_{[A]} \}$ be a representation for $G/(H, B)$ such that $Q_{[A]} \neq 0$ for $[A] \neq [\emptyset]$. If $\pi_{T,Q} : C^*(G/(H, B)) \to C^*(T_e, Q_{[A]})$ is the $*$-homomorphism satisfying $\pi_{T,Q}(t_e) = T_e$, $\pi_{T,Q}(q_{[A]}) = Q_{[A]}$, and there is a strongly continuous action $\beta$ of $\mathbb{T}$ on $C^*(T_e, Q_{[A]})$ such that $\beta_z \circ \pi_{T,Q} = \pi_{T,Q} \circ \gamma_z$ for every $z \in \mathbb{T}$, then $\pi_{T,Q}$ is faithful.

**Proof.** Select an increasing sequence $\{ F_n \}$ of finite subsets of $\Phi_{sg}(G^0) \cup \Phi(G^1)$ such that $\bigcup_{n=1}^{\infty} F_n = \Phi_{sg}(G^0) \cup \Phi(G^1)$. For each $n$, Corollary 4.3 gives an isomorphism

$$\pi_n : C^*(G_{F_n}) \to C^*(\{ t_e, q_{[v]} : [v] \in F^0, e \in F^1 \})$$

that respects the generators. We can apply the gauge invariant uniqueness theorem for graph $C^*$-algebras to see that the homomorphism

$$\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \to C^*(T_e, Q_{[A]})$$

is faithful. Hence, for every $F_n$, the restriction of $\pi_{T,Q}$ on the $*$-subalgebra of $C^*(G/(H, B))$ generated by $\{ t_e, q_{[v]} : [v] \in F^0_n, e \in F^1_n \}$ is faithful. This turns out that $\pi_{T,Q}$ is injective on the $*$-subalgebra $C^*(t_e, q_{[v]} : [v] \in \Phi_{sg}(G^0), e \in \Phi(G^1))$. Since, this subalgebra is dense in $C^*(G/(H, B))$, we conclude that $\pi_{T,Q}$ is faithful. □

**Proposition 4.6.** Let $G$ be an ultragraph. If $(H, B)$ is an admissible pair in $G$, then $C^*(G/(H, B)) \cong C^*(G)/I_{(H, B)}$. 
Proof. Using Proposition 3.3, we can consider $I_{H,B}$ as an ideal of $C^*(\mathcal{G})$. Suppose that $C^*(\mathcal{G}) = C^*(s_c,p_A)$ and $C^*(\mathcal{G}/(H,B)) = C^*(t_e,q_{[A]})$. If we define 

$$T_e := s_c + I_{H,B} \text{ and } Q_{[A]} := p_A + I_{H,B}$$

for every $[A] \in \Phi(G^0)$ and $e \in \Phi(G^1)$, then the family $\{T_e, Q_{[A]}\}$ is a representation for $G/(H,B)$ in $C^*(\mathcal{G})/I_{H,B}$. So, there is a $*$-homomorphism $\phi : C^*(\mathcal{G}/(H,B)) \to C^*(\mathcal{G})/I_{H,B})$ such that $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$. Moreover, all $Q_{[A]}$ with $[A] \neq [\emptyset]$ are nonzero because $p_A + I_{H,B} = I_{H,B}$ implies $A \in H$. Then, an application of Theorem 4.5 yields that $\phi$ is faithful.

On the other hand, the family $\{T_e, Q_{[A]} : [A] \in \Phi(G^0), e \in \Phi(G^1)\}$ generates the quotient $C^*(\mathcal{G})/I_{H,B}$, and hence, $\phi$ is surjective as well. Therefore, $\phi$ is an isomorphism and the result follows. \qed

To prove a version of Cuntz-Krieger uniqueness theorem, we extend Condition (L) for quotient ultragraphs.

**Definition 4.7.** We say that $G/(H,B)$ satisfies Condition (L) if for every loop $\alpha = e_1 \cdots e_n$ in $G/(H,B)$, at least one of the following conditions holds:

(i) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$ (or equivalently, $r(e_i) \setminus s(e_{i+1}) \neq \emptyset$).

(ii) $\alpha$ has an exit; that means, there exists $f \in \Phi(G^1)$ such that $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$.

**Lemma 4.8.** Let $F$ be a finite subset of $\Phi_{sg}(G^0) \cup \Phi(G^1)$. If $G/(H,B)$ satisfies Condition (L), then so does the graph $G_F$.

**Proof.** Suppose that $G/(H,B)$ satisfies Condition (L). As the elements of $F^0 \cup \Gamma$ are sinks in $G_F$, every loop in $G_F$ is of the form $\tilde{\alpha} = (e_1, e_2) \cdots (e_n, e_1)$ corresponding with a loop $\alpha = e_1 \cdots e_n$ in $G/(H,B)$. So, fix a loop $\tilde{\alpha} = (e_1, e_2) \cdots (e_n, e_1)$ in $G_F$. Then $\alpha = e_1 \cdots e_n$ is a loop in $G/(H,B)$ and by Condition (L), one of the following holds:

(i) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$, or

(ii) there exists $f \in \Phi(G^1)$ such that $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$.

We can suppose in the case (i) that $s(e_{i+1}) \subseteq r(e_i)$ and $r(e_i)$ emits only the edge $e_{i+1}$ in $G/(H,B)$. Then, by the definition of $\Gamma$, there exists either $[v] \in F^0$ with $[v] \subseteq r(e_i) \setminus s(e_{i+1})$, or $\omega \in \Gamma$ with $\omega_i = 1$. Thus, either $(e_i, [v])$ or $(e_i, \omega)$ is an exit for the loop $\tilde{\alpha}$ in $G_F$, respectively.

Now assume case (ii) holds. If $f \in F^1$, then $(e_i, f)$ is an exit for $\tilde{\alpha}$. If $f \notin F^1$, for $[v] := s(f)$ we have either $[v] \notin F^0$ or

$$\exists \omega \in \Gamma \text{ with } \omega_i = 1 \text{ such that } [v] \subseteq R(\omega).$$

Hence, $(e_i, [v])$ or $(e_i, \omega)$ is an exit for $\tilde{\alpha}$, respectively. Consequently, in any case, $\tilde{\alpha}$ has an exit. \qed
Theorem 4.9 (The Cuntz-Krieger Uniqueness Theorem). Suppose that \( \mathcal{G}/(H,B) \) is a quotient ultragraph satisfying Condition (L). If \( \{ T_e, Q_A \} \) is a Cuntz-Krieger representation for \( \mathcal{G}/(H,B) \) in which all the projection \( Q_A \) are nonzero for \( [A] \neq [0] \), then the *-homomorphism \( \pi_{T,Q} : C^*(\mathcal{G}/(H,B)) \rightarrow C^*(T_e, Q_{[A]}) \) with \( \pi_{T,Q}(t_e) = T_e \) and \( \pi_{T,Q}(q_{[A]}) = Q_{[A]} \) is an isometrically isomorphism.

Proof. It suffices to show that \( \pi_{T,Q} \) is faithful. Similar to Theorem 4.5, choose an increasing sequence \( \{ F_n \} \) of finite sets such that \( \bigcup_{n=1}^{\infty} F_n = \Phi_{sg}(G^0) \cup \Phi(G^1) \).

By Corollary 4.3, there are isomorphisms \( \pi_n : C^*(G_{F_n}) \rightarrow C^*(\{ t_e, q_{[v]} : [v] \in F_n, e \in F_n^1 \}) \) that respect the generators. Since all the graphs \( G_{F_n} \) satisfy Condition (L) by Lemma 4.8, the Cuntz-Krieger uniqueness theorem for graph \( C^* \)-algebras implies that the *-homomorphisms

\[
\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \rightarrow C^*(T_e, Q_{[A]})
\]

are faithful. Therefore, \( \pi_{T,Q} \) is faithful on the subalgebra \( C^*(t_e, q_{[v]} : [v] \in \Phi_{sg}(G^0), e \in \Phi(G^1)) \) of \( C^*(\mathcal{G}/(H,B)) \). Since this subalgebra is dense in \( C^*(\mathcal{G}/(H,B)) \), we conclude that \( \pi_{T,Q} \) is a faithful homomorphism. \( \square \)

5. Primitive ideals in \( C^*(\mathcal{G}) \)

In this section, we apply quotient ultragraphs to describe primitive gauge invariant ideals of an ultragraph \( C^* \)-algebra. Recall that since every ultragraph \( C^* \)-algebra \( C^*(\mathcal{G}) \) is separable (as assumed \( G^0 \) to be countable), a prime ideal of \( C^*(\mathcal{G}) \) is primitive and vice versa [3, Corollaire 1].

To prove Proposition 5.4 below, we need the following simple lemmas.

Lemma 5.1. Let \( \mathcal{G}/(H,B) = (\Phi(G^0), \Phi(G^0), \Phi(G^1), r, s) \) be a quotient ultragraph of \( \mathcal{G} \). If \( \mathcal{G}/(H,B) \) does not satisfy Condition (L), then \( C^*(\mathcal{G}/(H,B)) \) contains an ideal Morita-equivalent to \( C(T) \).

Proof. Suppose that \( \gamma = e_1 \cdots e_n \) is a loop in \( \mathcal{G}/(H,B) \) without exits and \( r(e_i) = s(e_{i+1}) \) for \( 1 \leq i \leq n \). If \( C^*(\mathcal{G}/(H,B)) = C^*(t_e, q_{[A]}), \) for each \( i \) we have

\[
t_e^* t_e = q_{s(e_i)} = q_{s(e_{i+1})} = t_{e_{i+1}}^* t_{e_{i+1}}^*.
\]

Write \([v] := s(\gamma)\) and let \( L_{\gamma} \) be the ideal of \( C^*(\mathcal{G}/(H,B)) \) generated by \( q_{[v]} \).

Since \( \gamma \) has no exits in \( \mathcal{G}/(H,B) \) and we have

\[
q_{s(e_i)} = (t_{e_i} \cdots t_{e_{n-1}}) q_{[v]} (t_{e_n}^* \cdots t_{e_i}^*) \quad (1 \leq i \leq n),
\]

an easy argument shows that

\[
I_{\gamma} = \overline{\text{span}} \{ t^\alpha q_{[v]} t^\beta_{[\beta]} : \alpha, \beta \in (\mathcal{G}/(H,B))^*, [v] \subseteq r(\alpha) \cap r(\beta) \}.
\]

So, we get

\[
q_{[v]} I_{\gamma} q_{[v]} = \overline{\text{span}} \{ (t^\gamma)^n q_{[v]} (t^\gamma)^m : m, n \geq 0 \},
\]

where \( (t^\gamma)^0 := q_{[v]} \). We show that \( q_{[v]} I_{\gamma} q_{[v]} \) is a full corner in \( I_{\gamma} \) which is isometrically isomorphic to \( C(T) \). For this, let \( E \) be the graph with one vertex
w and one loop \( f \). If we set \( Q_w := q_v \) and \( T_f := t_\gamma q_v \), then \( \{ T_f, Q_w \} \) is a Cuntz-Krieger \( E \)-family in \( q_v I_q, q_v \). Assume \( C^*(E) = C^*(s_f, p_w) \). Since \( Q_w \neq 0 \), the gauge-invariant uniqueness theorem for graph \( C^* \)-algebras implies that the \( * \)-homomorphism \( \phi: C^*(E) \to q_v I_q, q_v \) with \( p_w \mapsto Q_w \) and \( s_f \mapsto T_f \) is faithful. Moreover, the \( C^* \)-algebra \( q_v I_q, q_v \) is generated by \( \{ T_f, Q_w \} \), and hence \( \phi \) is an isomorphism. As we know \( C^*(E) \cong C(T) \), \( q_v I_q, q_v \) is isomorphic to \( C(T) \). Moreover, since \( q_v \) generates \( I_q \), the corner \( q_v I_q, q_v \) is full in \( I_q \).

Thus, \( I_q \) is Morita-equivalent to \( q_v I_q, q_v \cong C(T) \) and the proof is complete. \( \square \)

**Lemma 5.2.** If \( \mathcal{G}/(H, B) \) satisfies Condition (L), then any nonzero ideal in \( C^*(\mathcal{G}/(H, B)) \) contains projection \( q_{[A]} \) for some \( [A] \neq [0] \).

**Proof.** Take an arbitrary ideal \( J \) in \( C^*(\mathcal{G}/(H, B)) \). If there are no \( q_{[A]} \in J \) with \([A] \neq [0] \), then Theorem 4.9 implies that the quotient homomorphism \( \phi: C^*(\mathcal{G}/(H, B)) \to C^*(\mathcal{G}/(H, B))/J \) is injective. Hence, we have \( J = \ker \phi = (0) \). \( \square \)

**Definition 5.3.** Let \( \mathcal{G} \) be an ultragraph. For two sets \( A, C \in \mathcal{G}^0 \), we write \( A \supseteq C \) if either \( A \supseteq C \), or there exists \( \alpha \in \mathcal{G}^* \) with \( \{\alpha\} \supseteq \{\gamma\} \) such that \( s(\alpha) \in A \) and \( C \subseteq r(\gamma) \). We simply write \( A \supseteq v, v \supseteq C \) if \( A \supseteq \{v\} \), \( \{v\} \supseteq \{w\} \), respectively. A subset \( M \subseteq \mathcal{G}^0 \) is said to be downward directed whenever for every \( A_1, A_2 \in M \), there exists \( \emptyset \neq C \subseteq M \) such that \( A_1, A_2 \supseteq C \).

**Proposition 5.4.** Let \( H \) be a saturated hereditary subset of \( \mathcal{G}^0 \). Then the ideal \( I_{(H, B_H)} \) in \( C^*(\mathcal{G}) \) is primitive if and only if the quotient ultragraph \( \mathcal{G}/(H, B_H) \) satisfies Condition (L) and the collection \( \mathcal{G}^0 \setminus H \) is downward directed.

**Proof.** Let \( I_{(H, B_H)} \) be a primitive ideal of \( C^*(\mathcal{G}) \). Since \( C^*(\mathcal{G})/I_{(H, B_H)} \cong C^*(\mathcal{G}/(H, B_H)) \), the zero ideal in \( C^*(\mathcal{G}/(H, B_H)) \) is primitive. If \( \mathcal{G}/(H, B_H) \) does not satisfy Condition (L), then \( C^*(\mathcal{G}/(H, B_H)) \) contains an ideal \( J \) Morita-equivalent to \( C(T) \) by Lemma 5.1. Select two ideals \( J_1, J_2 \) in \( C(T) \) with \( I_1 \cap I_2 = (0) \), and let \( J_1, J_2 \) be their corresponding ideals in \( J \). Then \( J_1 \) and \( J_2 \) are two nonzero ideals of \( C^*(\mathcal{G}/(H, B_H)) \) with \( J_1 \cap J_2 = (0) \), contradicting the primitness of \( C^*(\mathcal{G}/(H, B_H)) \). Therefore, \( \mathcal{G}/(H, B) \) satisfies Condition (L).

Now we show that \( M := \mathcal{G}^0 \setminus H \) is downward directed. For this, we take two arbitrary sets \( A_1, A_2 \in M \) and consider the ideals

\[
J_1 := C^*(\mathcal{G}/(H, B_H))q_{[A_1]}C^*(\mathcal{G}/(H, B_H))
\]

and

\[
J_2 := C^*(\mathcal{G}/(H, B_H))q_{[A_2]}C^*(\mathcal{G}/(H, B_H))
\]

in \( C^*(\mathcal{G}/(H, B_H)) \) generated by \( q_{[A_1]} \) and \( q_{[A_2]} \), respectively. Since \( A_1, A_2 \notin H \), the projections \( q_{[A_1]}, q_{[A_2]} \) are nonzero by Theorem 3.10, and so are the ideals \( J_1, J_2 \). The primitness of \( C^*(\mathcal{G}/(H, B_H)) \) implies that the ideal

\[
J_1J_2 = C^*(\mathcal{G}/(H, B_H))q_{[A_1]}C^*(\mathcal{G}/(H, B_H))q_{[A_2]}C^*(\mathcal{G}/(H, B_H))
\]
is nonzero, and hence \( q_{[A_1]} C^*(G/(H, B_H)) q_{[A_2]} \neq \{0\} \). As the set
\[
\text{span} \{ t_\alpha q_{[D]} t_\beta^* : \alpha, \beta \in (G/(H, B))^* \}, \quad r(\alpha) \cap [D] \cap r(\beta) \neq [0] \}
\]
is dense in \( C^*(G/(H, B_H)) \), there exist \( \alpha, \beta \in (G/(H, B_H))^* \) and \( [D] \in \Phi(G^0) \) such that \( q_{[A_1]} (t_\alpha q_{[D]} t_\beta^*) q_{[A_2]} \neq 0 \). In this case, we must have \( s(\alpha) \subseteq [A_1] \) and \( s(\beta) \subseteq [A_2] \) and thus, \( A_1, A_2 \geq C \) for \( C := r_G(\alpha) \cap D \cap r_G(\beta) \). Therefore, \( G^0 \setminus H \) is downward directed.

For the converse, we assume that \( G/(H, B_H) \) satisfies Condition (L) and the collection \( M = G^0 \setminus H \) is downward directed. Fix two nonzero ideals \( J_1, J_2 \) of \( C^*(G/(H, B_H)) \). By Lemma 5.2, there are nonzero projections \( q_{[A_1]} \in J_1 \) and \( q_{[A_2]} \in J_2 \). Then \( A_1, A_2 \notin H \) and, since \( M \) is downward directed, there exists \( C \in M \) such that \( A_1, A_2 \geq C \). Hence, the ideal \( J_1 \cap J_2 \) contains the nonzero projection \( q_G \). Since \( J_1 \) and \( J_2 \) were arbitrary, this concludes that the \( C^* \)-algebra \( C^*(G/(H, B_H)) \) is primitive and \( I_{(H, B_H)} \) is a primitive ideal in \( C^*(G) \) by Proposition 4.6.

The following proposition describes another kind of primitive ideals in \( C^*(G) \).

**Proposition 5.5.** Let \( (H, B) \) be an admissible pair in \( G \) and let \( B = B_H \setminus \{w\} \). Then the ideal \( I_{(H, B)} \) in \( C^*(G) \) is primitive if and only if \( A \geq w \) for all \( A \in G^0 \setminus H \).

**Proof.** Suppose that \( I_{(H, B)} \) is a primitive ideal and take an arbitrary \( A \in G^0 \setminus H \). If \( \overline{\mathbf{A}} := A \cup \{v' : v \in A \cap (B_H \setminus B)\} \), then \( q_{\overline{\mathbf{A}}} \) and \( q_{[w]} \) are two nonzero projections in \( C^*(G/(H, B)) \). If we consider ideals \( J_{\overline{\mathbf{A}}} := \langle q_{\overline{\mathbf{A}}} \rangle \) and \( J_{[w]} := \langle q_{[w]} \rangle \) in \( C^*(G/(H, B)) \), then the primitness of \( C^*(G/(H, B)) \) \( \cong C^*(G)/I_{(H, B)} \) implies that the ideal
\[
J_{\overline{\mathbf{A}}} J_{[w]} = C^*(G/(H, B)) q_{\overline{\mathbf{A}}} C^*(G/(H, B)) q_{[w]} C^*(G/(H, B))
\]
is nonzero, and hence \( q_{\overline{\mathbf{A}}} C^*(G/(H, B)) q_{[w]} \neq \{0\} \). So, there exist \( \alpha, \beta \in (G/(H, B))^* \) such that \( q_{\overline{\mathbf{A}}} t_\alpha t_\beta^* q_{[w]} \neq 0 \). Since \( [w'] \) is a sink in \( G/(H, B) \), we must have \( q_{\overline{\mathbf{A}}} t_\alpha t_{[w']} \neq 0 \). If \( |\alpha| = 0 \), then \( [w'] \subseteq [\overline{\mathbf{A}}] \), \( w' \in \overline{\mathbf{A}} \) and \( w \in A \).

If \( |\alpha| \geq 1 \), then \( s(\alpha) \subseteq [\overline{\mathbf{A}}] \) and \( [w'] \subseteq r(\alpha) \), which follow \( s_G(\alpha) \in A \) and \( w \in r_G(\alpha) \). Therefore, we obtain \( A \geq w \) in either case.

Conversely, assume \( A \geq w \) for every \( A \in G^0 \setminus H \). Then the collection \( G^0 \setminus H \) is downward directed. Moreover, for every \( [0] \neq [A] \in \Phi(G^0) \), there exists \( \alpha \in (G/(H, B))^* \) such that \( s(\alpha) \subseteq [A] \) and \( [w'] \subseteq r(\alpha) \). As \( [w] \) is a sink in \( G/(H, B) \), we see that the quotient ultragraph \( G/(H, B) \) satisfies Condition (L). Now similar to the proof of Proposition 5.4, we can show that \( I_{(H, B)} \) is a primitive ideal.

Recall that each loop in \( G/(H, B) \) comes from a loop in the initial ultragraph \( G \). So, to check Condition (L) for a quotient ultragraph \( G/(H, B) \), we can use the following.
Definition 5.6. Let $H$ be a saturated hereditary subset of $G^0$. For simplicity, we say that a path $\alpha = e_1 \cdots e_n$ lies in $G \setminus H$ whenever $r_\mathcal{G}(\alpha) \in G^0 \setminus H$. We also say that $\alpha$ has an exit in $G \setminus H$ if either $r_\mathcal{G}(e_i) \setminus s_\mathcal{G}(e_{i+1}) \in G^0 \setminus H$ for some $i$, or there is an edge $f$ with $r_\mathcal{G}(f) \in G^0 \setminus H$ such that $s(f) = s(e_i)$ and $f \neq e_i$, for some $1 \leq i \leq n$.

It is easy to verify that a quotient ultragraph $G/(H, B)$ satisfies Condition (L) if and only if every loop in $G \setminus H$ has an exit in $G \setminus H$. Hence we have:

**Theorem 5.7** (See [1, Theorem 4.7]). Let $G$ be an ultragraph. A gauge invariant ideal $I_{(H, B)}$ of $C^*(\mathcal{G})$ is primitive if and only if one of the following holds:

1. $B = B_H$, $G^0 \setminus H$ is downward directed, and every loop in $G \setminus H$ has an exit in $G \setminus H$.
2. $B = B_H \setminus \{w\}$ for some $w \in B_H$, and $A \supset w$ for all $A \in G^0 \setminus H$.

**Proof.** Let $I_{(H, B)}$ be a primitive ideal in $C^*(\mathcal{G})$. Then

$$C^*(G/(H, B)) \cong C^*(\mathcal{G})/I_{(H, B)}$$

is a primitive $C^*$-algebra. We claim that $|B_H \setminus B| \leq 1$. Indeed, if $w_1, w_2$ are two distinct vertices in $B_H \setminus B$, similar to the proof of Propositions 5.4 and 5.5, the primitivity of $C^*(G/(H, B))$ implies that the corner $q_{w_1}C^*(G/(H, B))q_{w_2}$ is nonzero. So, there exist $\alpha, \beta \in (G/(H, B))^\ast$ such that $q_{w_1}t_\alpha q_{w_2} \neq 0$. But we must have $|\alpha| = |\beta| = 0$ because $[w_1], [w_2]$ are two sinks in $G/(H, B)$. Hence, $q_{w_1}q_{w_2} \neq 0$ which is impossible because $q_{w_1}q_{w_2} = q_{w_1}q_{w_2} = q_{w_1}q_{w_2} = q_{\emptyset} = 0$. Thus, the claim holds. Now we may apply Propositions 5.4 and 5.5 to obtain the result. \hfill $\Box$

Following [10, Definition 7.1], we say that an ultragraph $G$ satisfies Condition (K) if every vertex $v \in G^0$ either is the base of no loops, or there are at least two loops $\alpha, \beta$ in $G$ based at $v$ such that neither $\alpha$ nor $\beta$ is a subpath of the other. In view of [10, Proposition 7.3], if $G$ satisfies Condition (K), then all ideals of $C^*(G)$ are of the form $I_{(H, B)}$. So, in this case, Theorem 5.7 describes all primitive ideals of $C^*(G)$.

6. Purely infinite ultragraph $C^*$-algebras via Fell bundles

Mark Tomforde in [17] determined ultragraph $C^*$-algebras in which every hereditary subalgebra contains infinite projections. Here, we consider the notion of “pure infiniteness” in the sense of Kirchberg-Rørdam [11], and generalize [8, Theorem 2.3] to ultragraph setting. In view of Proposition 3.14 and Theorem 4.16 of [11], a (not necessarily simple) $C^*$-algebra $A$ is purely infinite if and only if for every $a \in A^+ \setminus \{0\}$ and closed two-sided ideal $I \subseteq A$, $a + I$ in the quotient $A/I$ is either zero or infinite (in this case, $a$ is called properly infinite). Recall from [11, Definition 3.2] that an element $a \in A^+ \setminus \{0\}$ is called infinite if there is $b \in A^+ \setminus \{0\}$ such that $a \oplus b \subseteq a \oplus 0$ in the matrix algebra $M_2(A)$. 

So, the notion of pure infiniteness is directly related to the structure of ideals and quotients. In this section, we use the quotient ultragraphs to characterize purely infinite ultragraph $C^*$-algebras. Briefly, we consider the natural Z- grading (or Fell bundle) for $C^*(\mathcal{G})$ and then apply the results of [12, Section 4] for pure infiniteness of Fell bundles.

6.1. Condition (K) for $\mathcal{G}$

To prove the main result of this section, Theorem 6.6, we need to show that an ultragraph $\mathcal{G}$ satisfies Condition (K) if and only if every quotient ultragraph $\mathcal{G}/(H, B)$ satisfies Condition (L).

**Notation.** Let $\alpha = e_1 \cdots e_n$ be a path in an ultragraph $\mathcal{G}$. If $\beta = e_k e_{k+1} \cdots e_l$ is a subpath of $\alpha$, we simply write $\beta \subseteq \alpha$; otherwise, we write $\beta \nsubseteq \alpha$.

First, we show in the absence of Condition (K) for $\mathcal{G}$ that there is a quotient ultragraph $\mathcal{G}/(H, B)$ which does not satisfy Condition (L). For this, let $\mathcal{G}$ contain a loop $\gamma = e_1 \cdots e_n$ such that there are no loops $\alpha$ with $s(\alpha) = s(\gamma)$, $\alpha \nsubseteq \gamma$, and $\gamma \nsubseteq \alpha$. If $\gamma^0 := \{s_\mathcal{G}(e_1), \ldots, s_\mathcal{G}(e_n)\}$, define

$$X := \{r_\mathcal{G}(\alpha) \setminus \gamma^0 : \alpha \in \mathcal{G}^*, |\alpha| \geq 1, s_\mathcal{G}(\alpha) \in \gamma^0\},$$

$$Y := \left\{ \bigcup_{i=1}^n A_i : A_1, \ldots, A_n \in X, n \in \mathbb{N} \right\},$$

and set

$$H_0 := \{B \in \mathcal{G}^0 : B \subseteq A \text{ for some } A \in Y \}.$$

We construct a saturated hereditary subset $H$ of $\mathcal{G}^0$ as follows: for any $n \in \mathbb{N}$ inductively define

$$S_n := \{w \in \mathcal{G}^0 : 0 < |s_\mathcal{G}^{-1}(w)| < \infty \text{ and } r_\mathcal{G}(s_\mathcal{G}^{-1}(w)) \subseteq H_{n-1} \}$$

and

$$H_n := \{ A \cup F : A \in H_{n-1} \text{ and } F \subseteq S_n \text{ is a finite subset} \}.$$

Then we can see that the subset

$$H = \bigcup_{n=0}^{\infty} H_n = \left\{ A \cup F : A \in H_0 \text{ and } F \subseteq \bigcup_{n=1}^{\infty} S_n \text{ is a finite subset} \right\}$$

is hereditary and saturated.

**Lemma 6.1.** Suppose that $\gamma = e_1 \cdots e_n$ is a loop in $\mathcal{G}$ such that there are no loops $\alpha$ with $s(\alpha) = s(\gamma)$ and $\alpha \nsubseteq \gamma$, $\gamma \nsubseteq \alpha$. If we construct the set $H$ as above, then $H$ is a saturated hereditary subset of $\mathcal{G}^0$. Moreover, we have $A \cap \gamma^0 = \emptyset$ for every $A \in H$.

**Proof.** By induction, we first show that each $H_n$ is a hereditary set in $\mathcal{G}$. For this, we check conditions (H1)-(H3) in Definition 2.5. To verify condition (H1) for $H_0$, let us take $e \in \mathcal{G}^1$ with $s_\mathcal{G}(e) \in H_0$. Then $s_\mathcal{G}(e) \in X$ and there is $\alpha \in \mathcal{G}^*$ such that $s_\mathcal{G}(\alpha) \in \gamma^0$ and $s_\mathcal{G}(e) \in r_\mathcal{G}(\alpha) \setminus \gamma^0$. Hence, $s_\mathcal{G}(ae) = s_\mathcal{G}(\alpha) \in \gamma^0$. Moreover, we have $r_\mathcal{G}(ae) \cap \gamma^0 = \emptyset$ because the otherwise implies the existence
of a path $\beta \in G^*$ with $s_G(\beta) = s_G(\gamma)$ and $\beta \nsubseteq \gamma$, $\gamma \nsubseteq \beta$, contradicting the hypothesis. It turns out

$$r_G(e) = r_G(\alpha e) = r_G(\alpha e) \setminus \gamma^0 \in X \subseteq H_0.$$ 

Hence, $H_0$ satisfies condition (H1). We may easily verify conditions (H2) and (H3) for $H_0$, so $H_0$ is hereditary. Moreover, for every $w \in S_n$, the range of each edge emitted by $w$ belongs to $H_{n-1}$ by definition. Thus, we can inductively check that each $H_n$ is hereditary, and so is $H = \cup_{n=1}^{\infty} H_n$. The saturation property of $H$ may be verified similar to the proof of [17, Lemma 3.12].

It remains to show $A \cap \gamma^0 = \emptyset$ for every $A \in H$. To do this, note that $A \cap \gamma^0 = \emptyset$ for every $A \in H_0$ because this property holds for all $A \in X$. We claim that $(\cup_{n=1}^{\infty} S_n) \cap \gamma^0 = \emptyset$. Indeed, if $v = s_G(e_i) \in \gamma^0$ for some $e_i \in \gamma$, then $r_G(e_i) \cap \gamma^0 \neq \emptyset$ and $r_G(e_i) \notin H_0$. Hence, $\{r_G(e) : e \in G^1, s_G(e) = v\} \not\subseteq H_0$ that turns out $v \notin S_1$. So, we have $S_1 \cap \gamma^0 = \emptyset$. An inductive argument shows $S_n \cap \gamma^0 = \emptyset$ for $n \geq 1$, and the claim holds. Now since

$$H = \cup_{n=1}^{\infty} H_n = \{A \cup F : A \in H_0 \text{ and } F \subseteq \cup_{n=1}^{\infty} S_n \text{ is a finite subset}\},$$

we conclude that $A \cap \gamma^0 = \emptyset$ for all $A \in H$. \hfill $\square$

**Proposition 6.2.** An ultragraph $G$ satisfies Condition (K) if and only if for every admissible pair $(H, B)$ in $G$, the quotient ultragraph $G/(H, B)$ satisfies Condition (L).

**Proof.** Suppose that $G$ satisfies Condition (K) and $(H, B)$ is an admissible pair in $G$. Let $\alpha = e_1 \cdots e_n$ be a loop in $G/(H, B)$. Since $\alpha$ is also a loop in $G$, there is a loop $\beta = f_1 \cdots f_m$ in $G$ with $s_G(\alpha) = s_G(\beta)$, and neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$. Without loss of generality, assume $e_1 \neq f_1$. By the fact $s_G(\alpha) = s_G(\beta) \in r_G(\beta)$, we have $r_G(\beta) \notin H$, and so $r_G(f_1) \notin H$ by the hereditary property of $H$. Therefore, $f_1$ is an exit for $\alpha$ in $G/(H, B)$ and we conclude that $G/(H, B)$ satisfies Condition (L).

For the converse, suppose on the contrary that $G$ does not satisfy Condition (K). Then there exists a loop $\gamma = e_1 \cdots e_n$ in $G$ such that there are no loops $\alpha$ with $s(\alpha) = s(\gamma)$, $\alpha \nsubseteq \gamma$, and $\gamma \nsubseteq \alpha$. As Lemma 6.1, construct a saturated hereditary subset $H$ of $G^0$ and consider the quotient ultragraph $G/(H, B_H) = (\Phi(G^0), \Phi(G^0), \Phi(G^1), r, s)$. We show that $\gamma$ as a loop in $G/(H, B_H)$ has no exits and $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n$. If $f$ is an exit for $\gamma$ in $G/(H, B_H)$ such that $s(f) = s(e_i)$ and $f \neq e_j$, then $r_G(f) \notin H$ and $r_G(f) \cap \gamma^0 \neq \emptyset$ (if $r_G(f) \cap \gamma^0 = \emptyset$, then $r_G(f) = r_G(f) \setminus \gamma^0 \in X \subseteq H$, a contradiction). So, there is $e_i \in \gamma$ such that $s_G(e_i) \in r_G(f)$. If we set $\alpha := e_1 \cdots e_{j-1} f e_j \cdots e_n$, then $\alpha$ is a loop in $G$ with $s_G(\alpha) = s_G(\gamma)$, and $\alpha \nsubseteq \gamma$, $\gamma \nsubseteq \alpha$, that contradicts the hypothesis. Therefore, $\gamma$ has no exits in $G/(H, B_H)$. Moreover, we have $r(e_i) \cap \gamma^0 = s(e_{i+1})$ for each $1 \leq i \leq n$, because the otherwise gives an exit for $\gamma$ in $G/(H, B_H)$ by the construction of $H$. Hence,

$$r(e_i) \setminus s(e_{i+1}) = r(e_i) \setminus [\gamma^0] = \emptyset,$$
and we get \( r(e_i) = s(e_{i+1}) \) (note that the fact \( r_G(e_i) \setminus \gamma^0 \in H \) implies \( r(e_i) \setminus \gamma^0 = [r_G(e_i) \setminus \gamma^0] = [\emptyset] \)). Therefore, the quotient ultragraph \( G/(H, B_H) \) does not satisfy Condition \((L)\) as desired. \(\square\)

### 6.2. Purely infinite ultragraph \( C^* \)-algebras via Fell bundles

Every quotient ultragraph (or ultragraph) \( C^* \)-algebra 
\[
C^*(G/(H, B)) = C^*(q_{[A]}, t_e)
\]
is equipped with a natural \( \mathbb{Z} \)-grading or Fell bundle \( B = \{ B_n : n \in \mathbb{Z} \} \) with the fibers 
\[
B_n := \text{span} \{ t_{\mu q_{[A]}} t^*_{\nu} : \mu, \nu \in (G/(H, B))^*, |\mu| - |\nu| = n \}.
\]
These Fell bundles will be considered in this section. The fiber \( B_0 \) is the fixed point \( C^* \)-subalgebra of \( C^*(G/(H, B)) \) for the gauge action which is an AF \( C^* \)-algebra. An application of the gauge invariant uniqueness theorem implies that \( C^*(G/(H, B)) \) is isomorphic to the cross sectional \( C^* \)-algebra \( C^*_r(B) \) (we refer the reader to [5] for details about Fell bundles and their \( C^* \)-algebras). Moreover, since \( \mathbb{Z} \) is an amenable group, combining Theorem 20.7 and Proposition 20.2 of [5] implies that \( C^*(G/(H, B)) \) is also isomorphic to the reduced cross sectional \( C^* \)-algebra \( C^*_r(B) \).

Following [4, Definition 2.1], an \emph{ideal} in a Fell bundle \( B = \{ B_n \} \) is a family \( \mathcal{J} = \{ J_n \}_{n \in \mathbb{Z}} \) of closed subspaces \( J_n \subseteq B_n \), such that \( B_mB_n \subseteq J_{mn} \) and \( J_nB_m \subseteq J_{nm} \) for all \( m, n \in \mathbb{Z} \). If \( \mathcal{J} \) is an ideal of \( B \), then the family \( B/\mathcal{J} := \{ B_n/J_n \}_{n \in \mathbb{Z}} \) is equipped with a natural Fell bundle structure, which is called a \emph{quotient Fell bundle} of \( B \); cf. [5, Definition 21.14].

**Definition 6.3** ([12, Definition 4.1]). Let \( G/(H, B) \) be a quotient ultragraph and \( B = \{ B_n \}_{n \in \mathbb{Z}} \) is the above Fell bundle in \( C^*(G/(H, B)) \). We say that \( B \) is \emph{aperiodic} if for each \( n \in \mathbb{Z} \setminus \{ 0 \} \), each \( b_n \in B_n \), and every hereditary subalgebra \( A \) of \( B_0 \), we have 
\[
\inf \{ \|ab_n a\| : a \in A^+, \|a\| = 1 \} = 0.
\]
Furthermore, \( B \) is called \emph{residually aperiodic} whenever the quotient Fell bundle \( B/\mathcal{J} \) is aperiodic for every ideal \( \mathcal{J} \) of \( B \).

The following lemma is analogous to [12, Proposition 7.3] for quotient ultragraphs.

**Lemma 6.4.** Let \( G/(H, B) \) be a quotient ultragraph and let \( B = \{ B_n \}_{n \in \mathbb{Z}} \) be the Fell bundle associated to \( C^*(G/(H, B)) \). Then \( B \) is aperiodic if and only if \( G/(H, B) \) satisfies Condition \((L)\).

**Proof.** We may modify the proof of [12, Proposition 7.3] for our case by replacing elements \( s_\alpha s_{\beta}^* \) and \( s_\mu s_{\mu}^* \) with \( t_{\alpha q_{[A]}} t^*_{\beta} \) and \( t_{\mu q_{[A]}} t^*_{\mu} \), respectively. Then the proof goes along the same lines as the one in [12, Proposition 7.3]. \(\square\)
Corollary 6.5. Let $\mathcal{G}$ be an ultragraph and let $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}}$ be the described Fell bundle of $C^* (\mathcal{G})$. If $\mathcal{G}$ satisfies Condition (K), then $\mathcal{B}$ is residually aperiodic.

Proof. Suppose that $\mathcal{G}$ satisfies Condition (K). In view of [10, Proposition 7.3], we know that all ideals of $C^* (\mathcal{G})$ are graded and of the form $I (H, B)$. So, each ideal $\mathcal{J} = \{J_n\}_{n \in \mathbb{Z}}$ of $\mathcal{B}$ is corresponding with an ideal $I (H, B)$ with the homogenous components $J_n := I (H, B) \cap B_n$. Moreover, the quotient Fell bundle $\mathcal{B} / \mathcal{J} := \{B_n / J_n : n \in \mathbb{Z}\}$ is a grading (or a Fell bundle) for $C^* (\mathcal{G}) / I (H, B) \cong C^* (\mathcal{G} / (H, B))$. Therefore, quotient Fell bundles $\mathcal{B} / \mathcal{J}$ are corresponding with quotient ultragraphs $\mathcal{G} / (H, B)$. Since such quotient ultragraphs satisfy Condition (L) by Proposition 6.2, Lemma 6.4 follows the result.

Theorem 6.6. Let $\mathcal{G}$ be an ultragraph. Then $C^* (\mathcal{G})$ is purely infinite (in the sense of [11]) if and only if $\mathcal{G}$ satisfies Condition (K), and for every saturated hereditary subset $H$ of $\mathcal{G}$, we have

\begin{enumerate}
\item $B_H = \emptyset$, and
\item every $A \in \mathcal{G}^0 \setminus H$ connects to a loop $\alpha$ in $\mathcal{G} \setminus H$, which means $A \geq s_{\mathcal{G}} (\alpha)$ (see Definition 5.3).
\end{enumerate}

Proof. First, suppose that $C^* (\mathcal{G})$ is purely infinite. If $\mathcal{G}$ does not satisfy Condition (K), by the second paragraph in the proof of Proposition 6.2, there is a quotient ultragraph $\mathcal{G} / (H, B) / I (H, B)$ containing a loop $\alpha \in (\mathcal{G} / (H, B))^*$ with no exits in $\mathcal{G} / (H, B)$. The argument of Lemma 5.1 follows that the ideal $J := \langle q_{\alpha} (\alpha) \rangle \subseteq C^* (\mathcal{G} / (H, B))$ is Morita-equivalent to $C (\mathbb{T})$. Hence, the projection $p_{\alpha} (\alpha)$ is not properly infinite which contradicts [11, Theorem 4.16].

Now assume that $H$ is a saturated hereditary subset of $\mathcal{G}$. We consider the quotient ultragraph $\mathcal{G} / (H, \emptyset)$ and take an arbitrary $[A] \in \Phi (\mathcal{G}^0) \setminus \{[\emptyset]\}$. If there is no loops $\alpha \in \mathcal{G}^{-1} (\mathcal{G}^0 \setminus H)$ with $A \geq s_{\mathcal{G}} (\alpha)$, then the ideal $I_A := \langle q_{[A]} \rangle \subseteq C^* (\mathcal{G} / (H, \emptyset))$ is AF. Thus $q_{[A]}$ is not infinite and $C^* (\mathcal{G})$ contains a non-properly infinite projection, contradicting [11, Theorem 4.16]. Moreover, we notice that for any $w \in B_H$, $[w^*]$ is a sink in $\mathcal{G} / (H, \emptyset)$ and the projection $q_{[w^*]}$ is not infinite, which is impossible.

Conversely, suppose that $\mathcal{G}$ satisfies Condition (K) and the asserted properties hold for any saturated hereditary set $H$. To show that $C^* (\mathcal{G})$ is purely infinite we apply [12, Theorem 5.12] for the pure infiniteness of Fell bundles. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}}$ be the natural Fell bundle in $C^* (\mathcal{G})$. Corollary 6.5 says that $\mathcal{B}$ is residually aperiodic. Moreover, every projection in $B_0$ is Murray-von Neumann equivalent to a finite sum $\sum_{i=1}^n r_i s_{\alpha_i} p_B \tilde{s}_{\beta_i}$ of mutually orthogonal projections such that $|\alpha_i| = |\beta_i|$ for $1 \leq i \leq n$. Note that each projection $s_{\alpha_i} p_B \tilde{s}_{\beta_i}$ is Murray-von Neumann equivalent to $(s_{\alpha_i} p_B)^* (p_B s_{\beta_i})$ which equals to zero unless $\alpha_i = \beta_i$. Hence, in view of [12, Lemma 5.13], it suffices to show that every nonzero projection of the form $s_{\mu} p_B \tilde{s}_{\mu}$ is properly infinite.

Let $I (H, \emptyset)$ be an ideal in $C^* (\mathcal{G})$ such that $s_{\mu} p_B \tilde{s}_{\mu} \notin I (H, \emptyset)$. Then $B \cap r_\mathcal{G} (\mu) \in \mathcal{G}^0 \setminus H$. Assume $C^* (\mathcal{G} / (H, \emptyset)) = C^* (\mathcal{T}, q_{[A]})$ and let $q : C^* (\mathcal{G}) \rightarrow C^* (\mathcal{G} / (H, \emptyset))$
be the canonical quotient map by Proposition 4.6. Then \(q(s_\mu B s^*_\mu) = t_\mu q|B| t^*_\mu\) \(\neq 0\). By hypothesis, there is a path \(\lambda\) and a loop \(\alpha \in r_G^{-1}(G \setminus H)\) such that \(s_G(\lambda) \in B \cap r_G(\mu)\) and \(s_G(\alpha) \in r_G(\lambda)\). Since \(\mathcal{G}\) satisfies Condition \((K)\), \(\alpha\) has an exit \(f\) in \(r^{-1}(G \setminus H)\). Thus we have
\[
(t_\alpha q_\alpha(\alpha))^* (t_\alpha q_\alpha(\alpha))^* + t_f t_f^* \leq q_\alpha(\alpha),
\]
and since
\[
(t_\alpha q_\alpha(\alpha))^* \sim (t_\alpha q_\alpha(\alpha))^* (t_\alpha q_\alpha(\alpha))^* = q_\alpha(\alpha),
\]
it turns out that \(q_\alpha(\alpha)\) is an infinite projection in \(C^*(\mathcal{G}/(H, \emptyset)) \cong C^*(\mathcal{G})/I_{(H, \emptyset)}\). On the other hand, the fact
\[
(t_{\mu \lambda} q_\beta(\beta))^* t_\mu q|B| t^*_\mu (t_{\mu \lambda} q_\beta(\beta)) = q_\beta(\beta)
\]
says that \(q_\beta(\beta) \lesssim t_\mu q|B| t^*_\mu\) (see [15, Proposition 2.4]), and thus \(t_\mu q|B| t^*_\mu\) is infinite by [11, Lemma 3.17]. It follows that \(s_{\mu B} B^*_\mu\) is a properly infinite projection. Now apply [12, Theorem 5.11(ii)] to conclude that the \(C^*\)-algebra \(C^*(\mathcal{G}) \cong C^*_\mu(B)\) is purely infinite. □

References


Hossein Larki
Department of Mathematics
Faculty of Mathematical Sciences and Computer
Shahid Chamran University of Ahvaz
P.O. Box: 83151-61357
Ahvaz, Iran
Email address: h.larki@scu.ac.ir