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# $\phi$-prime Subsemimodules of Semimodules over Commutative Semirings

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Abstract. Let $R$ be a commutative semiring with identity and $M$ be a unitary $R$-semimodule. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function, where $\mathcal{S}(M)$ is the set of all subsemimodules of $M$. A proper subsemimodule $N$ of $M$ is called $\phi$-prime subsemimodule, if $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$ implies that $r \in (N :_R M)$ or $x \in N$. So if we take $\phi(N) = \varnothing$ (resp., $\phi(N) = \{0\}$), a $\phi$-prime subsemimodule is prime (resp., weakly prime). In this article we study the properties of several generalizations of prime subsemimodules.

## 1. Introduction

Anderson and Bataineh [3] have introduced the concept of $\phi$-prime ideals in a commutative ring as a generalization of weakly prime ideals in a commutative ring introduced by Anderson and Smith [2]. After that several authors [1, 4, 9, 14, 15], etc. explored this concept in different ways either in commutative ring or semiring theory. In this article, we have extended the results of prime ideals of commutative ring to prime subsemimodules of semimodules.

For the definitions of monoid, semiring, semimodule and subsemimodule of a semimodule we refer [6]. All semirings in this paper are commutative with non-zero identity. The semiring $R$ is to be also a semimodule over itself. In this case, the subsemimodules of $R$ are called ideals of $R$. Let $M$ be a semimodule over a semiring $R$. A subtractive subsemimodule (= $k$-subsemimodule) $N$ is a subsemimodule of $M$ such that if $x,\ x + y \in N$, $y \in M$, then $y \in N$. If $N$ is a proper subsemimodule of an $R$-semimodule $M$, then we denote $(N :_R M) = \{r \in R : rM \subseteq N\}$ and $\sqrt{(N :_R M)} = \{r \in R : r^n M \subseteq N \text{ for some n} \in \mathbb{N}\}$. We say that $N$ is a radical submodule of $M$ if $\sqrt{(N :_R M)} = (N :_R M)$.

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## 2. $\phi$-prime Subsemimodules

In this section, we introduce the concept of a $\phi$-prime subsemimodule of a semimodule $M$ over a commutative semiring $R$ and prove some results related to it.

**Definition 2.1.** Let $\mathcal{S}(M)$ be the set of subsemimodule of $M$ and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. The proper subsemimodule $N$ of $M$ is called a *$\phi$-prime semimodule* if $r \in R$, $x \in M$ and $rx \in N \setminus \phi(N)$, then $r \in (N :_R M)$ or $x \in N$.

Since $N \setminus \phi(N) = N \setminus (N \cap \phi(N))$, we will assume throughout this article that $\phi(N) \subseteq N$. In the rest of the article we use the following functions $\phi_\alpha : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$.

$$
\begin{aligned}
\phi_\emptyset(N) &= \emptyset, \quad \forall N \in \mathcal{S}(M),\\
\phi_0(N) &= \{0\}, \quad \forall N \in \mathcal{S}(M),\\
\phi_1(N) &= (N :_R M)N, \quad \forall N \in \mathcal{S}(M),\\
\phi_2(N) &= (N :_R M)^2 N, \quad \forall N \in \mathcal{S}(M),\\
\phi_w(N) &= \cap_{i=1}^{\infty} (N :_R M)^i N, \quad \forall N \in \mathcal{S}(M).
\end{aligned}
$$

Then it is clear that $\phi_\emptyset$ and $\phi_0$-prime subsemimodules are prime and weakly prime subsemimodules respectively. Evidently for any subsemimodule and every positive integer $n \geqslant 2$, we have the following implications:

$$prime \Rightarrow \phi_w - prime \Rightarrow \phi_n - prime \Rightarrow \phi_{n-1} - prime.$$

For functions $\phi, \psi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$, we write $\phi \leqslant \psi$ if $\phi(N) \subseteq \psi(N)$ for each $N \in \mathcal{S}(M)$. So whenever $\phi \leqslant \psi$, any $\phi$-prime subsemimodule is $\psi$-prime.

**Definition 2.2.** A proper subsemimodule $N$ of $M$ is called *$M$-subtractive* if $N$ and $\phi(N)$ are subtractive subsemimodules of $M$.

The following propositions asserts that under some conditions $\phi$-prime subsemimodules are prime.

**Proposition 2.3.** *If $N$ is a $\phi$-prime subsemimodule of $M$ and $\phi(N)$ is a prime subsemimodule, then $N$ is a prime subsemimodule of $M$.*

*Proof.* Let $r \in R$, $x \in M$ and $rx \in N$. If $rx \notin \phi(N)$, since $N$ is a $\phi$-prime subsemimodule, then $x \in N$ or $r \in (N :_R M)$, and if $rx \in \phi(N)$, then $x \in \phi(N) \subseteq N$ or $r \in (\phi(N) :_R M) \subseteq (N :_R M)$. □

**Proposition 2.4.** *Let $R$ be a semiring and $M$ be an $R$-semimodule. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function and $N$ be a $\phi$-prime $M$-subtractive subsemimodule of $M$ such that $(N :_R M)N \nsubseteq \phi(N)$. Then $N$ is a prime subsemimodule of $M$.*

*Proof.* Let $a \in R$ and $x \in M$ be such that $ax \in N$. If $ax \notin \phi(N)$, then since $N$ is $\phi$-prime, we have $a \in (N :_R M)$ or $x \in N$.

So let $ax \in \phi(N)$. In this case we may assume that $aN \subseteq \phi(N)$. For, let $aN \nsubseteq \phi(N)$. Then there exists $p \in N$ such that $ap \notin \phi(N)$, so that $a(x+p) \in N \backslash \phi(N)$. Therefore $a \in (N :_R M)$ or $x + p \in N$ and hence $a \in (N :_R M)$ or $x \in N$. Second we may assume that $(N :_R M)x \subseteq \phi(N)$. If this is not the case, there exists $u \in (N :_R M)$ such that $ux \notin \phi(N)$ and so $(a+u)x \in N\backslash\phi(N)$. Since $N$ is a $\phi$-prime subsemimodule, we have $a+u \in (N :_R M)$ or $x \in N$. By [7, Lemma 1.2], $(N :_R M)$ is a subtractive ideal, and hence either $a \in (N :_R M)$ or $x \in N$. Now since $(N :_R M)N \nsubseteq \phi(N)$, there exist $r \in (N :_R M)$ and $p \in N$ such that $rp \notin \phi(N)$. So $(a+r)(x+p) \in N\backslash\phi(N)$, and hence $a+r \in (N :_R M)$ or $x+p \in N$. Therefore $a \in (N :_R M)$ or $x \in N$, since $N$ is a subtractive subsemimodule and by [7, Lemma 1.2], $(N :_R M)$ is a subtractive ideal. Hence $N$ is a prime subsemimodule of $M$. □

**Corollary 2.5.** *Let $P$ be a weakly prime subtractive subsemimodule of $M$ such that $(P :_R M)P \neq 0$. Then $P$ is a prime subsemimodule of $M$.*

*Proof.* In Proposition 2.4 set $\phi = \phi_0$. □

**Corollary 2.6.** *Let $M$ be an $R$-semimodule and $P$ be a $\phi$-prime $M$-subtractive subsemimodule of $M$. Then*

(i) $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)}$ *or* $\sqrt{(\phi(P) :_R M)} \subseteq (P :_R M)$.

(ii) *If* $(P :_R M) \subsetneq \sqrt{(\phi(P) :_R M)}$ *then $P$ is not a prime subsemimodule of $M$.*

(iii) *If* $\sqrt{(\phi(P) :_R M)} \subsetneq (P :_R M)$, *then $P$ is a prime subsemimodule of $M$.*

(iv) *If $\phi(P)$ is a radical subsemimodule of $M$, either $(P :_R M) = (\phi(P) :_R M)$ or $P$ is a prime subsemimodule of $M$.*

*Proof.* (i) If $P$ is not a prime subsemimodule of $M$, then by Proposition 2.4, we have $(P :_R M)P \subseteq \phi(P)$. Hence

$$\sqrt{(P :_R M)^2} \subseteq \sqrt{((P :_R M)P :_R M)} \subseteq \sqrt{(\phi(P) :_R M)}.$$

So $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)}$. If $P$ is a prime subsemimodule of $M$, then by [12, Lemma 3.3], $\sqrt{(\phi(P) :_R M)} \subseteq \sqrt{(P :_R M)} = (P :_R M)$ (note that we may assume that $\phi(P) \subseteq P$), and all the claims of the corollary follows.
(ii) Suppose $P$ is a prime subsemimodule of $M$. Then by $(i)$ we have, $\sqrt{(\phi(P) :_R M)} \subseteq (P :_R M)$. Also by assumption $(P :_R M) \subsetneq \sqrt{(\phi(P) :_R M)}$. Thus $\sqrt{(\phi(P) :_R M)} \subseteq (P :_R M) \subsetneq \sqrt{(\phi(P) :_R M)}$, which is a contradiction.
(iii) Suppose $P$ is not a prime subsemimodule of $M$. Then by $(i)$ we have, $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)}$. Also by assumption $\sqrt{(\phi(P) :_R M)} \subsetneq (P :_R M)$. Thus $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)} \subsetneq (P :_R M)$, which is a contradiction.
(iv) Suppose that $P$ is not prime subsemimodule. Then by $(i)$ and definition of radical subsemimodule we have

$$(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)} = (\phi(P) :_R M) \subseteq (P :_R M),$$

so $(P :_R M) = (\phi(P) :_R M)$. □

Let $M$ be an $R$-semimodule and $N$ a subsemimodule of $M$. For every $r \in R, \{m \in M |\ rm \in N\}$ is denoted by $(N :_M r)$. It is easy to see that $(N :_M r)$ is a subsemimodule of $M$ containing $N$.

**Theorem 2.7.** *Let $N$ be a $M$-subtractive subsemimodule of $M$ and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function. Then the following are equivalent:*

(i) *$N$ is a $\phi$-prime subsemimodule of $M$;*

(ii) *for $x \in M \setminus N$, $(N :_R x) = (N :_R M) \cup (\phi(N) :_R x)$;*

(iii) *for $x \in M \setminus N$, $(N :_R x) = (N :_R M)$ or $(N :_R x) = (\phi(N) :_R x)$;*

(iv) *for any ideal $I$ of $R$ and any subsemimodule $L$ of $M$, if $IL \subseteq N$ and $IL \nsubseteq \phi(N)$, then $I \subseteq (N :_R M)$ or $L \subseteq N$;*

(v) *for any $r \in R \setminus (N :_R M)$, $(N :_M r) = N \cup (\phi(N) :_M r)$;*

(vi) *for any $r \in R \setminus (N :_R M)$, $(N :_M r) = N$ or $(N :_M r) = (\phi(N) :_M r)$.*

*Proof.* (i) ⇒ (ii). Let $x \in M \setminus N$, $a \in (N :_R x) \setminus (\phi(N) :_R x)$. Then $ax \in N \setminus \phi(N)$. Since $N$ is a $\phi$-prime subsemimodule of $M$, So $a \in (N :_R M)$. As we may assume that $\phi(N) \subseteq N$, the other inclusion always holds.
(ii) ⇒ (iii). It is follows by [7, Lemma 1.2] and [16, Lemma 6].
(iii) ⇒ (iv). Let $I$ be an ideal of $R$ and $L$ be a subsemimodule of $M$ such that $IL \subseteq N$. Suppose $I \nsubseteq (N :_R M)$ and $L \nsubseteq N$. We show that $IL \subseteq \phi(N)$. Let $a \in I$ and $x \in L$. First let $a \notin (N :_R M)$. Then, since $ax \in N$, we have $(N :_R x) \neq (N :_R M)$. Hence by our assumption $(N :_R x) = (\phi(N) :_R x)$. So $ax \in \phi(N)$. Now assume that $a \in I \cap (N :_R M)$. Let $u \in I \setminus (N :_R M)$. Then $a + u \in I \setminus (N :_R M)$. So by the first case, for each $x \in L$ we have $ux \in \phi(N)$ and $(a+u)x \in \phi(N)$. This gives that $ax \in \phi(N)$. Thus in any case $ax \in \phi(N)$, since $N$ is $M$-subtractive. Therefore $IL \subseteq \phi(N)$.
(iv) ⇒ (i). Let $ax \in N \setminus \phi(N)$. Then $(a)(x) \subseteq N \setminus \phi(N)$. By considering the ideal $(a)$ and the submodule $(x)$, the result follows.
(i) ⇒ (v) Suppose $N$ is a $\phi$-prime subsemimodule such that $r \notin (N :_R M)$. Let $m \in (N :_M r)$. If $rm \notin \phi(N)$, then $N$ is $\phi$-prime implies $m \in N$, if $rm \in \phi(N)$, then $m \in (\phi(N) :_M r)$. Hence $(N :_M r) \subseteq N \cup (\phi(N) :_M r)$. The other containment holds trivially.
(v) ⇒ (vi) Let $(N :_M r) = N \cup (\phi(N) :_M r)$ for $r \in R \setminus (N :_R M)$. Then by [10, Lemma 3.13] and [5, Theorem 6], either $(N :_M r) = N$ or $(N :_M r) = (\phi(N) :_M r)$. Therefore, the result follows.
(vi) ⇒ (i) Let $r \in R \setminus (N :_R M)$, $m \in M$ such that $rm \in N \setminus \phi(N)$. By assumption, either $(N :_M r) = N$ or $(N :_M r) = (\phi(N) :_M r)$. As $rm \notin \phi(N)$, then $m \notin (\phi(N) :_M r)$ and as $m \in (N :_M r)$, we have $(N :_M r) \neq (\phi(N) :_M r)$. Hence $(N :_M r) = N$ and so $m \in N$ as required. □

Recall from [13, Definition 2] that, an $R$-subsemimodule $N$ of $M$ is said to be a strong subsemimodule if for each $x \in N$, there exists $y \in N$ such that $x + y = 0$.

**Theorem 2.8.** *Let $f : M \to M'$ be an epimorphism of $R$-semimodules with $f(0) = 0$ and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{S}(M') \to \mathcal{S}(M') \cup \{\emptyset\}$ be two functions. Then the following statements hold:*

(i) *If $N'$ is a $\phi'$-prime subsemimodule of $M'$ and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a $\phi$-prime subsemimodule of $M$.*

(ii) *If $N$ is a strong $\phi$-prime subsemimodule of $M$ containing $Ker(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a $\phi'$-prime subsemimodule of $M'$.*

*Proof.* (i) Since $f$ is epimorphism, $f^{-1}(N')$ is a proper subsemimodule of $M$. Let $a \in R$ and $m \in M$ such that $am \in f^{-1}(N') \setminus \phi(f^{-1}(N'))$. Since $am \in f^{-1}(N')$, so $af(m) \in N'$. Also, $af(m) \notin \phi'(N')$ as $\phi(f^{-1}(N)) = f^{-1}(\phi'(N'))$. Therefore $af(m) \in N' \setminus \phi'(N')$, and so either $a \in (N' : M')$ or $f(m) \in N'$. Hence either $aM' \subseteq N'$, or $m \in f^{-1}(N')$. Therefore either $af^{-1}(M') \subseteq f^{-1}(N')$, or $m \in f^{-1}(N')$. Thus, either $a \in (f^{-1}(N') : M)$ or $m \in f^{-1}(N')$. It follows that $f^{-1}(N')$ is a $\phi$-prime subsemimodule of $M$.
(ii) Let $ax \in f(N) \setminus \phi'(f(N))$ for some $a \in R$ and $x \in M'$. Since $ax \in f(N)$, there exists an element $x' \in N$ such that $ax = f(x')$. Since $f$ is epimorphism and $x \in M'$, then there exists $y \in M$ such that $f(y) = x$. As $x' \in N$ and $N$ is a strong subsemimodule of $M$, there exists $x^{''} \in N$ such that $x' + x^{''} = 0$, which gives $f(x' + x^{''}) = 0$. Therefore, $ax + f(x^{''}) = 0$ or $f(ay) + f(x^{''}) = 0$ which implies that $ay + x^{''} \in Kerf \subseteq N$. Thus $ay \in N$, since $N$ is a substractive subsemimodule. Since $ax \notin \phi'(f(N))$, one can see easily that $ay \notin \phi(N)$. So $ay \in N \setminus \phi(N)$, and hence either $a \in (N : M)$ or $y \in N$, as $N$ is $\phi$-prime. Thus, either $aM \subseteq N$, or $f(y) \in f(N)$, and hence either $af(M) \subseteq f(N)$, or $x \in f(N)$. Thus, $a \in (f(N) : M')$ or $x \in f(N)$. It follows that $f(N)$ is a $\phi$-prime subsemimodule of $M'$. □

The notion of fractions of semimodule $M$ was defined by Atani in [11]. Also it show that there is a one-to-one correspondence between the set of all prime subsemimodules $N$ of $M$ with $(N : M) \cap S = \varnothing$ and the set of all prime subsemimodules of the $R_P$-semimodule $M_P$, see [11, Corollary 2]. In the next theorem we want to generalize this fact for $\phi$-prime subsemimodules. Let $S$ be a multiplicatively closed subset of $R$ and $N(S) = \{x \in M : \exists s \in S, sx \in N\}$. We know that $N(S)$ is a subsemimodule of $M$ containing $N$ and $S^{-1}(N(S)) = S^{-1}N$. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function and define $\phi_S : \mathcal{S}(S^{-1}M) \to \mathcal{S}(S^{-1}M) \cup \{\varnothing\}$ by $\phi_S(S^{-1}N) = S^{-1}(\phi(N(S))$ if $\phi(N(S)) \neq \varnothing$, and $\phi_S(S^{-1}N) = \varnothing$ if $\phi(N(S)) = \varnothing$. Since $\phi(N) \subseteq N$, $\phi_S(S^{-1}N) \subseteq S^{-1}N$.

**Lemma 2.9.** *If $N$ is a subtractive subsemimodule of $M$, so is $S^{-1}N \cap M$.*

*Proof.* Let $a, a + b \in S^{-1}N \cap M$. Then $a/1 = c_1/s_1$ and $(a + b)/1 = c_2/s_2$, such that $c_1, c_2 \in N$ and $s_1, s_2 \in S$. So $t_2(a + b)s_2 = t_2c_2 \in N$ and $t_2t_1as_1 = t_2t_1c_1 \in N$ for some $t_1, t_2 \in S$. Consequently $t_1t_2as_1s_2 + t_1t_2bs_1s_2 = t_1t_2s_1c_2 \in N$. Since $N$

is subtractive, therefore $t_1t_2bs_1s_2 \in N$. Thus $b/1 = (t_1t_2s_1s_2b)/(t_1t_2s_1s_2) \in S^{-1}N$, and so $b \in S^{-1}N \cap M$. Hence $S^{-1}N \cap M$ is subtractive. □

**Theorem 2.10.** *Let $N$ be a $\phi$-prime subsemimodule of $M$. Let $S$ be a multiplicatively closed subset of $R$ such that $S^{-1}N \neq S^{-1}M$ and $S^{-1}(\phi(N)) \subseteq \phi_S(S^{-1}N)$. Then $S^{-1}N$ is a $\phi_S$-prime subsemimodule of $S^{-1}M$. Furthermore if $N$ is a $M$-subtractive subsemimodules of $M$ and $S^{-1}N \neq S^{-1}(\phi(N))$, then $N = N(S) = S^{-1}N \cap M$.*

*Proof.* First suppose that there is an element $s$ which is common to $(N :_R M)$ and $S$. So, $sM \subseteq N$. Suppose now that $m/s' \in S^{-1}M$. Then $m/s' = sm/ss' \in S^{-1}N$. This show that $S^{-1}N = S^{-1}M$ and the first assertion follows. From here on we assume that $(N :_R M) \cap S = \emptyset$ and $N$ be a $\phi$-prime subsemimodule of $M$. Let $(a_1/s_1)(a_2/s_2) \in S^{-1}N \setminus \phi_S(S^{-1}N)$ for some $a_1 \in R$, $a_2 \in M$, $s_1, s_2 \in S$. Then there exist $c \in N$ and $s' \in S$ such that $a_1a_2/s_1s_2 = c/s'$. Then $us'a_1a_2 = us_1s_2c \in N$ for some $u \in S$. If $us'a_1a_2 \in \phi(N)$, then $(a_1/s_1)(a_2/s_2) = (us'a_1a_2)/(us's_1s_2) \in S^{-1}(\phi(N)) \subseteq \phi_S(S^{-1}N)$, a contradiction. Hence $us'a_1a_2 \in N - \phi(N)$. As $N$ is a $\phi$-prime subsemimodule, we get $us'a_1 \in (N :_R M)$ or $a_2 \in N$. It follows that $a_1/s_1 = (us'a_1)/(us's_1) \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$, or $a_2/s_2 \in S^{-1}N$. Consequently $S^{-1}N$ is a $\phi_S$-prime subsemimodule of $S^{-1}M$.

To prove the last part of the theorem, assume that $N$ be $M$-subtractive subsemimodules of $M$ and $S^{-1}N \neq S^{-1}(\phi(N))$. Clearly, $N \subseteq S^{-1}N \cap M$. For the reverse containment, pick an element $a \in S^{-1}N \cap M$. Then there exist $c \in N$ and $s \in S$ with $a/1 = c/s$. Therefore, $tsa = tc$ for some $t \in S$. If $(ts)a \notin \phi(N)$, then $(ts)a \in N \setminus \phi(N)$ and $(N :_R M) \cap S = \emptyset$ gives $a \in N$. So assume that $(ts)a \in \phi(N)$. In this case $a \in S^{-1}(\phi(N)) \cap M$. Therefore, $S^{-1}N \cap M \subseteq N \cup (S^{-1}(\phi(N)) \cap M)$. It follows that either $S^{-1}N \cap M = S^{-1}(\phi(N)) \cap M$ or $S^{-1}N \cap M = N$ by Lemma 2.9 and [5, Theorem 6]. If the former case holds, then by [14, Lemma 2.5], $S^{-1}N = S^{-1}(\phi(N))$ which is a contradiction. So the result follows. □

We say that $r \in R$ is a zero-divisor of a semimodule $M$ if $rm = 0$ for some non-zero element $m$ of $M$. The set of all zero-divisors of $M$ is denoted by $Z_R(M)$ (see [11, Definition1 (f)]).

**Proposition 2.11.** *Let $S^{-1}N$ be a $\phi_S$-prime subsemimodule of $S^{-1}M$ such that $S^{-1}(\phi(N)) = \phi_S(S^{-1}N)$, $S \cap Z_R(M/N) = \emptyset$ and $S \cap Z_R(N/\phi(N)) = \emptyset$. Then $N$ is a $\phi$-prime subsemimodule of $M$.*

*Proof.* Let $a \in R$, $m \in M$, $am \in N \setminus \phi(N)$ and $m \notin N$. Then $(a/1)(m/1) = am/1 \in S^{-1}N$. If $(a/1)(m/1) \in \phi_S(S^{-1}N) = S^{-1}(\phi(N))$, then there exists $s \in S$ such that $sam \in \phi(N)$ which contradicts $S \cap Z_R(N/\phi(N)) = \emptyset$. Therefore $(a/1)(m/1) \in S^{-1}N \setminus \phi_S(S^{-1}N)$, and so either $a/1 \in (S^{-1}N : S^{-1}M)$ or $m/1 \in S^{-1}N$. If $m/1 \in S^{-1}N$, then $sm \in N$ for some $s \in S$, and so $s \in S \cap Z_R(M/N)$ which is a contradiction. Thus $a/1 \in (S^{-1}N : S^{-1}M)$. Hence for every $c \in M$, $sac \in N$ and since $s \in S$, we have $s \in Z_R(M/N)$. Since $S \cap Z_R(M/N) = \emptyset$, it follows that $ac \in N$, and so $a \in (N :_R M)$. □

A subsemimodule $N$ of an $R$-semimodule $M$ is called partitioning subsemimodule (= $Q$-subsemimodule) if there exists a subset $Q$ of $M$ such that

(1) $M = \cup\{q + N : q \in Q\}$, and

(2) $(q_1 + N) \cap (q_2 + N) \neq \varnothing$, for any $q_1, q_2 \in Q$ if and only if $q_1 = q_2$.

Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$, and $M/N_{(Q)} = \{q + N : q \in Q\}$. Then $M/N_{(Q)}$ forms an $R$-semimodule under the binary operations $\oplus$ and $\odot$, which are defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + N$, where $r \in R$ and $q_4 \in Q$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. Then, this $R$-semimodule $M/N_{(Q)}$ is called the quotient semimodule of $M$ by $N$.

Moreover, if $N$ is a $Q$-subsemimodule of an $R$-semimodule $M$ and $L$ is a $k$-subsemimodule of $M$ containing $N$, then $N$ is a $Q \cap L$-subsemimodule of $L$ and $L/N = \{q + N : q \in Q \cap L\}$ is a subtractive subsemimodule of $M/N_{(Q)}$ as given by Chaudhari and Bonde in 2010, see [8].

Let $N$ be a $Q$-subsemimodule of $M$ and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function. We define $\phi_N : \mathcal{S}(M/N_{(Q)}) \to \mathcal{S}(M/N_{(Q)}) \cup \{\varnothing\}$ by $\phi_N(P/N) = (\phi(P))/N$ for every $P \in \mathcal{S}(M)$ with $N \subseteq \phi(P)$ where $P$ is a $M$-subtractive subsemimodule (and $\phi_N(P/N) = \varnothing$ if $\phi(P) = \varnothing$).

**Theorem 2.12.** *Let $N$ be a $Q$-subsemimodule of an $R$-semimodule $M$, $P$ a $M$-subtractive subsemimodule and $N$ a $\phi$-prime subsemimodule of $M$ with $N \subseteq \phi(P)$. Then $P$ is a $\phi$-prime subsemimodule of $M$ if and only if $P/N_{(Q\cap P)}$ is a $\phi$-prime subsemimodule of $M/N_{(Q)}$.*

*Proof.* Let $P$ be a $\phi$-prime subsemimodule of $M$. Let $r \in R, q_1 + N \in M/N_{(Q)}$ be such that $r \odot (q_1 + N) \in P/N_{(Q\cap P)} \setminus (\phi(P))/N$. By [8, Theorem 3.5], there exists a unique $q_2 \in Q \cap P$ such that $r \odot (q_1 + N) = q_2 + N$ where $rq_1 + N \subseteq q_2 + N$. Since $N \subseteq P, rq_1 \in P$.
If $rq_1 \in \phi(P)$, then $rq_1 \in (q_2 + N) \cap (q_0 + N)$ and $q_0 \in \phi(P)$. So $q_2 = q_0$ and hence $q_0 + N = q_2 + N$ a contradiction. Thus $rq_1 \notin \phi(P)$. As $P$ is $\phi$-prime subsemimodule, either $rM \subseteq P$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q \cap P$ and hence $q_1 + N \in P/N_{(Q\cap P)}$. Suppose $rM \subseteq P$. For $q + N \in M/N_{(Q)}$, let $r \odot (q + N) = q_3 + N$ where $q_3$ is a unique element of $Q$ such that $rq + N \subseteq q_3 + N$. Since $N \subseteq P$ and $P$ is a subtractive subsemimodule of $M$, $q_3 \in P$. Hence $q_3 \in Q \cap P$. Now $r \odot (q + N) = q_3 + N \in P/N_{(Q\cap P)}$ and hence $r \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. So $P/N_{(Q\cap P)}$ is a $\phi$-prime subsemimodule of $M/N_{(Q)}$.

Conversely, suppose that $N, P/N_{(Q\cap P)}$ are $\phi$-prime subsemimodules of $M$, $M/N_{(Q)}$ respectively. Let $rm \in P \setminus \phi(P)$ where $r \in R, m \in M$. If $rm \in N$, then we are through, since $rm \in N \setminus \phi(N)$ and $N$ is a $\phi$-prime subsemimodule of $M$. So suppose $rm \in P \setminus N$. By using [7, Lemma 3.6], there exists a unique $q_1 \in Q$ such that $m \in (q_1 + N)$ and $rm \in r \odot (q_1 + N) = q_2 + N$ where $q_2$ is a unique element of $Q$ such that $rq_1 + N \subseteq q_2 + N$. Now $rm \in P, rm \in q_2 + N$ implies $q_2 \in P$, as $P$ is a subtractive subsemimodule and $N \subseteq P$. Hence $r \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)} \setminus (\phi(P))/N$. As $P/N_{(Q\cap P)}$ is a $\phi$-prime subsemimodule, either $r \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$ or $q_1 + N \in P/N_{(Q\cap P)}$. If $q_1 + N \in P/N_{(Q\cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. So assume that $r \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Let $x \in M$. By using [7, Lemma 3.6], there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and $rx \in r \odot (q_3 + N) = q_4 + N$ where $q_4$ is a unique element of $Q$ such that $rq_3 + N \subseteq q_4 + N$. Now $q_4 + N = r \odot (q_3 + N) \in P/N_{(Q\cap P)}$ and hence $q_4 \in P$. As $rx \in q_4 + N$ and $N \subseteq P$ implies $rx \in P$. So $rM \subseteq P$. □

**Theorem 2.13.** *Let $R$ be semiring, $I$ a $Q$-ideal of $R$ and $P$ a subtractive ideal of $R$ with $I \subseteq P$. Then $P$ is a prime (weakly prime) ideal of $R$ if and only if $P/I_{(Q\cap P)}$ is a prime (weakly prime) ideal of $R/I_{(Q)}$.*

*Proof.* The proof is similar as in the proof of Theorem 2.12. □

# References


[1] A. Y. Darani, *On 2-absorbing and weakly 2-absorbing ideals of commutative semirings*, Kyungpook. Math. Journal, **52(1)**(2012), 91–97.

[2] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math., **29**(2003), 831-840.

[3] D. D. Anderson and M. Bataineh, *Generalizatins of prime ideals*, Comm. Algebra, **36**(2008), 686-696.

[4] F. Soheilnia, *On 2-absorbing and weakly 2-absorbing primary ideals of a commutative semiring*, Kyungpook. Math. Journal, **56(2)**(2016), 107–120.

[5] G. Yesilotr, K. H. Oral, *On prime subsemimodules of semimodules*, Int. J. Algebra, **4(1)**(2010), 53–60.

[6] J. S. Golan, *Semiring and their Applications*, Kluwer Academic publisher Dordrecht, 1999.

[7] J. N. Chaudhari and D. R. Bonde, *Weakly prime subsemimodules of semimodules over semirings*, Int. J. Algebra, **5(4)**(2011), 167–174.

[8] J. N. Chaudhari and D. R. Bonde, *On partitioning and subtractive subsemimodules of semimodules over semirings*, Kyungpook. Math. Journal, **50(2)**(2010), 329–336.

[9] R. Safakish, F. Fatahi, M. Lotfiparsa, *$\phi$-primary subtractive ideals in semirings*, Int. J. Pure Appl. Math., **108(3)**(2016), 629–633.

[10] R. P. Deore, *On associated primes and primary subsemimodule*, Int. J. Algebra, **2(16)**(2008), 795–801.

[11] R. Ebrahimi Atani, S. Ebrahimi Atani, *Spectra of semimodule*, Bull. Acad. Stiiunte. Rep. mold. matematica, **3(67)**(2011), 15–28.

[12] S. Ebrahimi Atani and M. Shajarian Kohan, *A note on finitely generated multiplication semimodules over commutative semirings*, Int. J. Algebra, **4(8)**(2010), 389–396.

[13] S. Ebrahimi Atani, R. Ebrahimi Atani, U. Tekir, *A Zariski topology for semimodules*, Eur. J. Pure Appl. Math., **4(3)**(2011), 251–265.

[14] M. Bataineh and R. Malas, *Primal and weakly primal subsemimodules*, American Int. J. Contemp. Research, **4(1)**(2014), 131–135.

[15] N. Zamani, *$\phi$-prime submodules*, Glasgow Math. J., **52(2)**(2010), 253–259.

[16] V. Gupta, J. N. Chaudhari, *Characterisation of weakly prime subtractive ideals in semirings*, Bulletin of the Insttiute of Math., **3**(2008), 347–352.