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# Global Existence and Ulam–Hyers Stability of Ψ–Hilfer Fractional Differential Equations

Kishor Deoman Kucche* and Jyoti Pramod Kharade
*Department of Mathematics, Shivaji University, Kolhapur-416 004, Maharashtra, India*
*e-mail* : kdkucche@gmail.com *and* jyoti.thorwe@gmail.com

Abstract. In this paper, we consider the Cauchy-type problem for a nonlinear differential equation involving a Ψ-Hilfer fractional derivative and prove the existence and uniqueness of solutions in the weighted space of functions. The Ulam–Hyers and Ulam–Hyers–Rassias stabilities of the Cauchy–type problem is investigated via the successive approximation method. Further, we investigate the dependence of solutions on the initial conditions and their uniqueness using $\epsilon$-approximated solutions. Finally, we present examples to illustrate our main results.

## 1. Introduction

The theory of fractional differential equations (FDEs) [9] and their applications is a topic of great interest in pure and applied mathematics. The popluarity of FDEs and related problems is largely due to the many applications they have to various branches of science and engineering. The varied applications have yielded many different definitions of fractional derivative and fractional integral, which do not coincide in general. Hilfer [7] introduced the generalized Riemann–Liouville fractional derivative of order $\mu$ ($n-1<\mu<n\in\mathbb{N}$) and of type $\nu$ ($0\leq\nu\leq 1$), defined by

$$\mathbf{D}_{a+}^{\mu,\nu}y\left(t\right)=\mathbf{I}_{a+}^{\nu(n-\mu)}\left(\frac{d}{dx}\right)^{n}\mathbf{I}_{a+}^{(1-\nu)(n-\mu)}y\left(t\right),$$

which allows one to interpolate between the Riemann-Liouville derivative $\mathbf{D}_{a+}^{\mu,0}={}^{RL}\mathbf{D}_{a+}^{\mu}$ and the Caputo derivative $\mathbf{D}_{a+}^{\mu,1}={}^{C}\mathbf{D}_{a+}^{\mu}$. Furati et al. [5, 6] considered the

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basic problems of existence, uniqueness and stability of solutions of the nonlinear Cauchy type problem involving a Hilfer fractional derivative.

Very recently, Sousa and Olivera [15] extended the concept of the Hilfer derivative operator and introduced a new definition of the fractional derivative– namely the $\Psi$–Hilfer fractional derivative of a function of order $\mu$ and of type $\nu$ with respect to another function $\Psi$. They discussed its calculus and derived a class of fractional integrals and fractional derivatives by giving a particular value to the function $\Psi$. In [16] Sousa and Olivera proved a generalized Gronwall inequality involving thier fractional integral with respect to another function and investigated basic results pertaining to the existence, uniqueness of solutions of, and and data dependence of, the Cauchy type problem involving a $\Psi$-Hilfer differential operator.

The fundamental problem of Ulam [14] was generalized for the stability of FDEs [18]. Stability of any FDE in the Ulam–Hyers sense is the problem of dealing with the replacement of a given FDE by a fractional differential inequality, and obtaining sufficient conditions about "When the solutions of the fractional differential inequalities are close to the solutions of given FDE ?". For a Ulam–Hyers stability theory of FDEs and its recent development, one can refer to [1, 2, 3, 17, 18] and the references therein.

Huang et al. [8] investigated HU stability of integer order delay differential equations by the method of successive approximation. Kucche and Sutar [10] extended the idea of [8] and investigated the HU stability of nonlinear delay FDEs with the Caputo derivative. Oliveira and Sousa [4, 17] explored Ulam–Hyers and Ulam–Hyers–Rassias stabilities of $\Psi$- Hilfer nonlinear fractional differential and integrodifferential equations by means of the fixed point theorem of Banach.

Motivated by the work of [10, 15, 16], in this paper, we consider the $\Psi$–Hilfer fractional differential equation ($\Psi$–Hilfer FDE) of the form:

$$ {}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y(t) = f(t, y(t)),\ t \in [a,b],\ 0<\mu<1,\ 0 \le \nu \le 1, \tag{1.1} $$

$$ \mathbf{I}_{a^+}^{1-\rho;\,\Psi} y(a) = y_a \in \mathbb{R},\ \rho = \mu+\nu-\mu\nu, \tag{1.2} $$

where ${}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi}(\cdot)$ is the (left-sided) $\Psi$-Hilfer fractional derivative of order $\mu$ and type $\nu$, $\mathbf{I}_{a^+}^{1-\rho;\,\Psi}$ is (left-sided) fractional integral of order $1-\rho$ with respect to another function $\Psi$, in the Riemann–Liouville sense, and $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a given function that will be specified latter.

The main objective of this paper is to prove the global existence and uniqueness of solutions to $\Psi$–Hilfer FDE (1.1)-(1.2). Using the method of successive approximations we investigate Ulam–Hyers (HU) and Ulam–Hyers–Rassias (HUR) stability of (1.1). Utilizing the generalized Gronwall inequality [15] we obtain estimations for the difference between two $\epsilon$-approximated solutions of (1.1)-(1.2). With this we derive the results pertaining to uniqueness and dependence of solutions on the initial conditions.

The $\Psi$–Hilfer FDE (1.1)-(1.2) is quite general in the sense that for different particular values of the parameters $\mu$, $\nu$ and for various specific functions $\Psi$ the derivative operator ${}^H\mathbf{D}_{a^+}^{\mu,\,\nu;\,\Psi}$ reduces to many well known fractional derivative operators that are recorded in [15]. Among these are the: Riemann–Liouville derivative, Caputo derivative, Hilfer derivative, Katugampola Derivative, Caputo-Katugampola Derivative, Hilfer-Katugampola Derivative, Hadmard Derivative, Caputo-Hadmard Derivative, Hilfer-Hadmard Derivative, Chen fractional derivative, Jumarie derivative, Prabhakar derivative, Erd'elyi-Kober derivative, Riesz derivative, Feller derivative, Weyl derivative, Cassar derivative, and Caputo-Riesz derivative.

Moreover, for $\Psi(t) = t$ and $\nu = 1$ the results of the current paper yield results from [10] and for $\Psi(t) = t$ and $\mu = \nu = 1$ yield results from [8].

The paper is organized as follows. In Section 2, some basic definitions and results concerning the Ψ–Hilfer fractional derivative that are important for the development of the paper are given. Section 3 deals with the existence and uniqueness of solutions of the problem (1.1)-(1.2). Section 4 deals with the HU stability of (1.1) via successive approximations. In Section 5, we study an $\epsilon-$approximate solution of (1.1). In Section 6, we provide an illustrative example.

## 2. Preliminaries

In this section, we recall few definitions, notions and the fundamental results about the fractional integrals of a function with respect to another function [9], and the Ψ–Hilfer fractional operator [15, 16].

Let $0 < a < b < \infty$, $\Delta = [a, b] \subset \mathbb{R}_+ = [0, \infty), 0 \le \rho < 1$ and $\Psi \in C^1(\Delta, \mathbb{R})$ be an increasing function such that $\Psi'(x) \neq 0,\ \forall\ x \in \Delta$. The weighted spaces $\mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$, $\mathbf{C}^{\rho}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ and $\mathbf{C}^{\mu,\nu}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ of functions are defined as follows:

(i) $\mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}) = \left\{h : (a, b] \to \mathbb{R} :\ (\Psi(t) - \Psi(a))^{1-\rho} h(t) \in \mathbf{C}(\Delta, \mathbb{R})\right\}$, with the norm

$$||h||_{\mathbf{C}_{1-\rho;\,\Psi}} = \max_{t\in\Delta} \left|(\Psi(t) - \Psi(a))^{1-\rho}\, h(t)\right|,$$

(ii) $\mathbf{C}^{\rho}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}) = \left\{h \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}) : \mathbf{D}^{\rho}_{a^+} h(t) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})\right\}$,

(iii) $\mathbf{C}^{\mu,\nu}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}) = \left\{h \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}) :\ {}^{H}\mathbf{D}^{\mu,\nu}_{a^+} h(t) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})\right\}$.

**Definition 2.1.**([9, 12]) The Ψ*–Riemann fractional integral* of order $\mu > 0$ of the function h is given by

$$\mathbf{I}^{\mu;\,\Psi}_{a+} h\,(t) := \frac{1}{\Gamma\,(\mu)} \int_a^t \mathcal{L}^{\mu}_{\Psi}(t, \eta) h\,(\eta)\ d\eta,$$

where

$$\mathcal{L}^{\mu}_{\Psi}(t, \eta) = \Psi'\,(\eta)\,(\Psi\,(t) - \Psi\,(\eta))^{\mu-1}\,.$$

**Lemma 2.2.** *Let $\mu > 0$, $\nu > 0$ and $\delta > 0$. Then*

(i) $\mathbf{I}^{\mu;\,\Psi}_{a^+}\mathbf{I}^{\nu;\,\Psi}_{a^+} h(t) = \mathbf{I}^{\mu+\nu;\,\Psi}_{a^+} h(t)$,

(ii) *if* $h(t) = (\Psi(t) - \Psi(a))^{\delta-1}$, *then* $\mathbf{I}^{\mu;\,\Psi}_{a^+} h(t) = \frac{\Gamma(\delta)}{\Gamma(\mu+\delta)} (\Psi(t) - \Psi(a))^{\mu+\delta-1}$.

We need the following results [9, 12] which are useful in the analysis of the paper.

**Lemma 2.3.**([16]) *If* $\mu > 0$ *and* $0 \leq \rho < 1$, *then* $\mathbf{I}_{a^+}^{\mu;\,\Psi}$ *is bounded from* $\mathbf{C}_{\rho;\,\Psi}(\Delta, \mathbb{R})$ *to* $\mathbf{C}_{\rho;\,\Psi}(\Delta, \mathbb{R})$. *Also, if* $\rho \leq \mu$, *then* $\mathbf{I}_{a^+}^{\mu;\,\Psi}$ *is bounded from* $\mathbf{C}_{\rho;\,\Psi}(\Delta, \mathbb{R})$ *to* $C(\Delta, \mathbb{R})$.

**Definition 2.4.**([15]) The $\Psi$*–Hilfer fractional derivative* of a function $h$ of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$, is defined by

$$^{H}\mathbf{D}_{a^+}^{\mu,\,\nu;\,\Psi} h(t) = \mathbf{I}_{a^+}^{\nu(1-\mu);\,\Psi} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right) \mathbf{I}_{a^+}^{(1-\nu)(1-\mu);\,\Psi} h(t).$$

**Lemma 2.5.**([15]) *If* $h \in C^1(\Delta, \mathbb{R})$, $0 < \mu < 1$ *and* $0 \leq \nu \leq 1$, *then*

(i) $\mathbf{I}_{a^+}^{\mu;\,\Psi}\, {}^{H}\mathbf{D}_{a^+}^{\mu,\,\nu;\,\Psi} h(t) = h(t) - \Omega_{\Psi}^{\rho}(t,a) \mathbf{I}_{a^+}^{(1-\nu)(1-\mu);\Psi} h(a)$, *where* $\Omega_{\Psi}^{\rho}(t,a) = \frac{(\Psi(t)-\Psi(a))^{\rho-1}}{\Gamma(\rho)}$.

(ii) ${}^{H}\mathbf{D}_{a^+}^{\mu,\,\nu;\,\Psi}\, \mathbf{I}_{a^+}^{\mu;\,\Psi} h(t) = h(t)$.

**Definition 2.6.**([9]) Let $\mu > 0, \nu > 0$. The *one parameter Mittag-Leffler function* is defined as

$$E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + 1)},$$

and the *two parameter Mittag-Leffler function* is defined as

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + \nu)}.$$

## 3. Existence and Uniqueness Results

In this section we derive the existence and uniqueness results of the Cauchy-type problem (1.1)-(1.2) by utilizing the following modified version of contraction principle.

**Lemma 3.1.**([13]) *Let* $\mathcal{X}$ *be a Banach space and let* $\mathcal{T}$ *be an operator which maps the element of* $\mathcal{X}$ *into itself for which* $\mathcal{T}^r$ *is a contraction, where* $r$ *is a positive integer then* $\mathcal{T}$ *has a unique fixed point.*

**Theorem 3.2.** *Let* $0 < \mu < 1$ *and* $0 \leq \nu \leq 1$, *and* $\rho = \mu + \nu - \mu\nu$. *Let* $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(\cdot, y(\cdot)) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ *for any* $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$, *and let* $f$ *satisfies the Lipschitz condition with respect to second argument*

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|, \tag{3.1}$$

*for all* $t \in (a, b]$ *and for all* $y_1, y_2 \in \mathbb{R}$, *where* $L > 0$ *is a Lipschitz constant. Then the Cauchy problem* (1.1)-(1.2) *has a unique solution in* $\mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$.

*Proof.* The equivalent fractional integral to the initial value problem (1.1)-(1.2) is given by [15]

$$
\begin{aligned}
y(t) &= \Omega_{\Psi}^{\rho}(t,a)\, y_a + \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,y(t)) \\
&= \Omega_{\Psi}^{\rho}(t,a)\, y_a + \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta) f(\eta, y(\eta))\, d\eta,\ t \in (a,b]. \qquad (3.2)
\end{aligned}
$$

Our aim is to prove that the fractional integral (3.2) has a solution in the weighted space $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. Consider the operator $\mathbb{T}$ defined on : $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ by

$$
(\mathbb{T}y)(t) = \Omega_{\Psi}^{\rho}(t,a)\, y_a + \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta) f(\eta, y(\eta))\, d\eta. \qquad (3.3)
$$

By Lemma 2.3, it follows that $\mathbf{I}_{a^+}^{\mu;\,\Psi} f(.,y(.)) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. Clearly, $y_a\, \Omega_{\Psi}^{\rho}(t,a) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. Therefore, from (3.3), we have $\mathbb{T}y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ for any $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. This proves $\mathbb{T}$ maps $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ into itself. Note that the fractional integral equation (3.2) can be written as fixed point operator equation

$$
y = \mathbb{T}y,\ \ y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R}).
$$

We prove that the above operator equation has a fixed point which will act as a solution for the problem (1.1)-(1.2). For any $t \in (a,b]$, consider the space $\mathbf{C}_{t;\,\Psi} = \mathbf{C}_{1-\rho;\,\Psi}([a,t],\mathbb{R})$ with the norm defined by,

$$
\|y\|_{\mathbf{C}_{t;\,\Psi}} = \max_{w\in[a,t]} \left| (\Psi(w)-\Psi(a))^{1-\rho} y(w) \right|.
$$

Using mathematical induction for any $y_1, y_2 \in \mathbf{C}_{t;\,\Psi}$ and $t \in (a,b]$, we prove that

$$
\|\mathbb{T}^j y_1 - \mathbb{T}^j y_2\|_{\mathbf{C}_{t;\,\Psi}} \le \Gamma(\rho) \frac{(L(\Psi(t)-\Psi(a))^{\mu})^j}{\Gamma(j\mu+\rho)} \|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}},\ \ j \in \mathbb{N}. \qquad (3.4)
$$

Let any $y_1, y_2 \in \mathbf{C}_{t;\,\Psi}$. Then from the definition of operator $\mathbb{T}$ given in (3.3) and using Lipschitz condition on $f$, we have

$$
\begin{aligned}
&\|\mathbb{T}y_1 - \mathbb{T}y_2\|_{\mathbf{C}_{t;\,\Psi}} \\
&= \max_{w\in[0,t]} \left| (\Psi(w)-\Psi(a))^{1-\rho} \left(\mathbb{T}y_1(w) - \mathbb{T}y_2(w)\right) \right| \\
&= \max_{w\in[0,t]} \left| (\Psi(w)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta) \left( f(\eta, y_1(\eta)) - f(\eta, y_2(\eta)) \right) d\eta \right| \\
&\le L \max_{w\in[0,t]} \left| (\Psi(w)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta) \left| y_1(\eta) - y_(\eta) \right| d\eta \right| \\
&= L \max_{w\in[0,t]} \left| (\Psi(t)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^w \left\{ \mathcal{L}_{\Psi}^{\mu}(w,\eta) (\Psi(\eta)-\Psi(a))^{\rho-1} \right\} \times \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \left\{(\Psi(\eta)-\Psi(a))^{1-\rho}\left|y_1(\eta)-y_2(\eta)\right|\right\} d\eta \,\Big| \\
&\le \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)} \int_a^t \mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1} \times \\
&\qquad \max_{w\in[0,\eta]} \left|(\Psi(w)-\Psi(a))^{1-\rho}(y_1(w)-y_2(w))\right| d\eta \\
&\le \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)} \|y_1-y_2\|_{c_{t;\,\Psi}} \int_a^t \mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\, d\eta \\
&\le L\|y_1-y_2\|_{\mathbf{C}_{t;\,\Psi}} \left[(\Psi(t)-\Psi(a))^{1-\rho}\, \mathbf{I}_{a^+}^{\mu;\,\Psi}(\Psi(t)-\Psi(a))^{\rho-1}\right] \\
&= \frac{(L(\Psi(t)-\Psi(a))^{\mu})\Gamma(\rho)}{\Gamma(\mu+\rho)} \|y_1-y_2\|_{\mathbf{C}_{t;\,\Psi}}.
\end{aligned}
$$

Thus the inequality (3.4) holds for $j=1$. Let us suppose that the inequality (3.4) holds for $j=r\in\mathbb{N}$, i.e. suppose

$$
\|\mathbb{T}^r y_1-\mathbb{T}^r y_2\|_{\mathbf{C}_{t;\,\Psi}} \le \Gamma(\rho)\frac{(L(\Psi(t)-\Psi(a))^{\mu})^r}{\Gamma(r\mu+\rho)}\|y_1-y_2\|_{\mathbf{C}_{t;\,\Psi}} \tag{3.5}
$$

holds. Next, we prove that (3.4) holds for $j=r+1$. Let $y_1, y_2\in \mathbf{C}_{t;\,\Psi}$ and denote $y_1^*=\mathbb{T}^r y_1$ and $y_2^*=\mathbb{T}^r y_2$. Then using the definition of operator $\mathbb{T}$ and Lipschitz condition on $f$, we get

$$
\begin{aligned}
&\|\mathbb{T}^{r+1}y_1-\mathbb{T}^{r+1}y_2\|_{\mathbf{C}_{t;\,\Psi}} \\
&= \|\mathbb{T}(\mathbb{T}^r y_1)-\mathbb{T}(\mathbb{T}^r y_2)\|_{\mathbf{C}_{t;\,\Psi}} \\
&= \|\mathbb{T}y_1^*-\mathbb{T}y_2^*\|_{\mathbf{C}_{t;\,\Psi}} \\
&= \max_{w\in[a,t]}\left|(\Psi(w)-\Psi(a))^{1-\rho}\left(\mathbb{T}y_1^*(w)-\mathbb{T}y_2^*(w)\right)\right| \\
&= \max_{w\in[a,t]}\left|(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \mathcal{L}_\Psi^\mu(w,\eta)\left(f(\eta,y_1^*(\eta))-f(\eta,y_2^*(\eta)\right)d\eta\right| \\
&\le L\max_{w\in[a,t]}\left\{(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \left(\mathcal{L}_\Psi^\mu(w,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\right)\times\right. \\
&\qquad \left.\left((\Psi(\eta)-\Psi(a))^{1-\rho}\left|y_1^*(\eta)-y_2^*(\eta)\right|\right) d\eta\right\} \\
&\le \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \left(\left\{\mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\right\}\times\right. \\
&\qquad \left.\max_{w\in[a,\eta]}\left|(\Psi(w)-\Psi(a))^{1-\rho}\left(y_1^*(w)-y_2^*(w)\right)\right|\right) d\eta \\
&\le \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\|y_1^*-y_2^*\|_{\mathbf{C}_{\eta;\,\Psi}}\, d\eta
\end{aligned}
$$

From (3.5), we have

$$\|y_1^* - y_2^*\|_{\mathbf{C}_{s;\,\Psi}} = \|\mathbb{T}^r y_1 - \mathbb{T}^r y_2\|_{\mathbf{C}_{s;\,\Psi}} \leq \Gamma(\rho)\frac{(L(\Psi(s)-\Psi(a))^{\mu})^r}{\Gamma(r\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{s;\,\Psi}}.$$

Therefore,

$$\begin{aligned}
&\|\mathbb{T}^{r+1} y_1 - \mathbb{T}^{r+1} y_2\|_{\mathbf{C}_{t;\,\Psi}} \\
&\leq \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\times \\
&\qquad \Gamma(\rho)\frac{(L(\Psi(\eta)-\Psi(a))^{\mu})^r}{\Gamma(r\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{\eta;\,\Psi}}\,d\eta \\
&\leq \left(\frac{L^{r+1}\Gamma(\rho)}{\Gamma(r\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}}\right)\times \\
&\qquad \left((\Psi(t)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)(\Psi(\eta)-\Psi(a))^{r\mu+\rho-1}\,d\eta\right) \\
&\leq \frac{L^{r+1}\Gamma(\rho)}{\Gamma(r\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}}(\Psi(t)-\Psi(a))^{1-\rho}\mathbf{I}_{a^+}^{\mu}(\Psi(t)-\Psi(a))^{r\mu+\rho-1} \\
&= \frac{L^{r+1}\Gamma(\rho)}{\Gamma(r\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}}(\Psi(t)-\Psi(a))^{1-\rho}\frac{\Gamma(r\mu+\rho)}{\Gamma((r+1)\mu+\rho)}\,(\Psi(t)-\Psi(a))^{(r+1)\mu+\rho-1} \\
&= \Gamma(\rho)\frac{(L(\Psi(t)-\Psi(a))^{\mu})^{r+1}}{\Gamma((r+1)\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}}.
\end{aligned}$$

Thus we have

$$\|\mathbb{T}^{r+1} y_1 - \mathbb{T}^{r+1} y_2\|_{\mathbf{C}_{t;\,\Psi}} \leq \Gamma(\rho)\frac{(L(\Psi(t)-\Psi(a))^{\mu})^{r+1}}{\Gamma((r+1)\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{t;\,\Psi}}.$$

Therefore, by principle of mathematical induction the inequality (3.4) holds for all $j \in \mathbb{N}$ and for every t in $\Delta$. As a consequence we find on the fundamental interval $\Delta$,

$$\|\mathbb{T}^j y_1 - \mathbb{T}^j y_2\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \leq \Gamma(\rho)\frac{(L(\Psi(b)-\Psi(a))^{\mu})^j}{\Gamma(j\mu+\rho)}\|y_1 - y_2\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}. \tag{3.6}$$

By definition of two parameter Mittag-Leffler function, we have

$$E_{\mu,\rho}(L(\Psi(b)-\Psi(a))^{\mu}) = \sum_{j=0}^{\infty}\frac{(L(\Psi(b)-\Psi(a))^{\mu})^j}{\Gamma(j\mu+\rho)}$$

Note that $\frac{(L(\Psi(b)-\Psi(a))^{\mu})^j}{\Gamma(j\mu+\rho)}$ is the $j^{th}$ term of the convergent series of real numbers. Therefore,

$$\lim_{j\to\infty}\frac{(L(\Psi(b)-\Psi(a))^{\mu})^j}{\Gamma(j\mu+\rho)} = 0.$$

Thus we can choose $j \in \mathbb{N}$ such that

$$\Gamma(\rho)\frac{(L(\Psi(b)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)} < 1,$$

so that $\mathbb{T}^j$ is a contraction.

Therefore, by Lemma 3.1, $\mathbb{T}$ has a unique fixed point $y^*$ in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$, which is a unique solution of the Cauchy type problem (1.1)–(1.2). □

**Remark 3.3.** The existence result proved above is with no restriction on the interval $\Delta = [a, b]$, and hence solution $y^*$ of (1.1)-(1.2) exists for any $a, b$ $(0 < a < b < \infty)$. Thus the Theorem 3.2 guarantees global unique solution in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$.

## 4. Ulam-Hyers Stability

To discuss HU and HUR stability of (1.1), we adopt the approach of [11, 18]. For $\epsilon > 0$ and continuous function $\phi : \Delta \to [0,\infty)$, we consider the following inequalities:

$$\left|{}^{H}\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t))\right| \le \epsilon,\ t \in \Delta, \tag{4.1}$$

$$\left|{}^{H}\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t))\right| \le \phi(t),\ t \in \Delta, \tag{4.2}$$

$$\left|{}^{H}\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t))\right| \le \epsilon\phi(t),\ t \in \Delta. \tag{4.3}$$

**Definition 4.1.** Problem (1.1) has *HU stability* if there exists a real number $\mathbf{C}_f > 0$ such that for each $\epsilon > 0$ and for each solution $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of the inequation (4.1) there exists a solution $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of (1.1) with

$$\|y^* - y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \le C_f\,\epsilon.$$

**Definition 4.2.** Problem (1.1) has *generalized HU stability* if there exists a function $C_f \in ([0,\infty)), [0,\infty))$ with $C_f(0) = 0$ such that for each solution $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of the inequation (4.1) there exists a solution $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of (1.1) with

$$\|y^* - y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \le C_f(\epsilon).$$

**Definition 4.3.** Problem (1.1) has *HUR stability* with respect to a function $\phi$ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of the inequation (4.3) there exists a solution $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of (1.1) with

$$|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t) - y(t))| \le C_{f,\phi}\,\epsilon\,\phi(t),\ t \in (\Delta,\mathbb{R}).$$

**Definition 4.4.** Problem (1.1) has *generalized HUR stabilit*y with respect to a function $\phi$ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $y^* \in$

$\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of the inequation (4.2) there exists a solution $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ of (1.1) with

$$|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t)-y(t))| \leq C_{f,\phi}\phi(t), \quad t \in \Delta.$$

In the next theorem we will make use of the successive approximation method to prove that the Ψ–Hilfer FDE (1.1) is HU stable.

**Theorem 4.5.** *Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ for any $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ and that satisfies the Lipschitz condition*

$$|f(t,y_1)-f(t,y_2)| \leq L|y_1-y_2|,\ t \in (a,b],\ y_1,y_2 \in \mathbb{R},$$

*where $L>0$ is a constant. For every $\epsilon > 0$, if $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ satisfies*

$$\left|{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t,y^*(t))\right| \leq \epsilon,\ t \in \Delta,$$

*then there exists a solution $y$ of equation (1.1) in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ with $\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) = \mathbf{I}_{a^+}^{1-\rho;\,\Psi} y(a)$ such that*

$$\|y^*-y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \leq \left(\frac{(E_\mu(L(\Psi(b)-\Psi(a))^\mu)-1)}{L}(\Psi(b)-\Psi(a))^{1-\rho}\right)\epsilon,\ t \in \Delta.$$

*Proof.* Fix any $\epsilon > 0$, let $z \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ satisfies

$$\left|{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t,y^*(t))\right| \leq \epsilon, \quad t \in \Delta. \tag{4.4}$$

Then there exists a function $\sigma_{y^*} \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ ( depending on $y^*$ ) such that $|\sigma_{y^*}(t)| \leq \epsilon,\ t \in \Delta$ and

$${}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) = f(t,y^*(t)) + \sigma_{y^*}(t),\ t \in \Delta. \tag{4.5}$$

If $y^*(t)$ satisfies (4.5) then it satisfies the equivalent fractional integral equation

$$\begin{aligned} y^*(t) &= \Omega_\Psi^\rho(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta) f(\eta,y^*(\eta))\,d\eta \\ &\quad + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta)\sigma_{y^*}(\eta)\,d\eta,\ t \in \Delta. \end{aligned} \tag{4.6}$$

Define

$$y_0(t) = y^*(t), \quad t \in \Delta \tag{4.7}$$

and consider the sequence $\{y_n\}_{n=1}^\infty \subseteq \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ defined by

$$y_n(t) = \Omega_\Psi^\rho(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta) f(\eta,y_{n-1}(\eta))\,d\eta,\ t \in \Delta. \tag{4.8}$$

Using mathematical induction firstly we prove that for every $t \in \Delta$ and $y_j \in \mathbf{C}_{1-\rho;\,\Psi}[a,t] = \mathbf{C}_{t;\,\Psi}$

$$\|y_j - y_{j-1}\|_{\mathbf{C}_{t;\,\Psi}} \le \frac{\epsilon}{L}\frac{(L(\Psi(t)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+1)}\left(\Psi(t)-\Psi(a)\right)^{1-\rho},\ j \in \mathbb{N}. \tag{4.9}$$

By definition of successive approximations and using (4.6) we have

$$\begin{aligned}
&\|y_1 - y_0\|_{\mathbf{C}_{t;\,\Psi}}\\
&= \max_{w\in[a,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\left\{y_1(w)-y_0(w)\right\}\right|\\
&= \max_{w\in[0,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\left(\Omega_{\Psi}^{\rho}(w,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}z(a) + \mathbf{I}_{a+}^{\mu;\,\Psi}f(w,y_0(w)) - y_0(w)\right)\right|\\
&= \max_{w\in[0,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\left(\Omega_{\Psi}^{\rho}(w,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}z(a) + \mathbf{I}_{a+}^{\mu;\,\Psi}f(w,z(w)) - z(w)\right)\right|\\
&= \max_{w\in[0,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta)\sigma_z(\eta)\,d\eta\right|\\
&\le \max_{w\in[0,t]} \left[(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta)|\sigma_z(\eta)|\,d\eta\right]\\
&\le \epsilon \max_{w\in[0,t]} \left[(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta)\,d\eta\right]\\
&\le \frac{\epsilon}{\Gamma(\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho}(\Psi(t)-\Psi(a))^{\mu}\\
&= \frac{\epsilon}{L}\frac{(L(\Psi(t)-\Psi(a))^{\mu})}{\Gamma(\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho},
\end{aligned}$$

Therefore,

$$\|y_1 - y_0\|_{\mathbf{C}_{t;\,\Psi}} \le \frac{\epsilon}{L}\frac{(L(\Psi(t)-\Psi(a))^{\mu})}{\Gamma(\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho},$$

which proves the inequality (4.9) for j = 1. Let us suppose that the inequality (4.9) holds for $j = r \in \mathbb{N}$, we prove it for $j = r+1$. By definition of successive approximations and Lipschitz condition on $f$, we obtain

$$\begin{aligned}
&\|y_{r+1} - y_r\|_{\mathbf{C}_{t;\,\Psi}}\\
&= \max_{w\in[0,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\left\{y_{r+1}(w)-y_r(w)\right\}\right|\\
&= \max_{w\in[0,t]} \left|(\Psi(w)-\Psi(a))^{1-\rho}\left(\mathbf{I}_{a+}^{\mu;\,\Psi}f(w,y_r(w)) - \mathbf{I}_{a+}^{\mu;\,\Psi}f(w,y_{r-1}(w))\right)\right|\\
&\le L \max_{w\in[0,t]} \left[(\Psi(w)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^w \mathcal{L}_{\Psi}^{\mu}(w,\eta)\ |y_r(\eta)-y_{r-1}(\eta)|\ d\eta\right]\\
&\le \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\times
\end{aligned}$$

$$\max_{w\in[0,\eta]}\left|(\Psi(w)-\Psi(a))^{1-\rho}\left\{y_r(w)-y_{r-1}(w)\right\}\right| d\eta$$
$$=\frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\|y_r-y_{r-1}\|_{\mathbf{C}_{\eta;\,\Psi}}\,d\eta.$$

Using the inequality (4.9) for j=r, we have

$$\begin{aligned}\|y_{r+1}-y_r\|_{\mathbf{C}_{t;\,\Psi}} &\leq \frac{L(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\times\\ &\qquad\left(\frac{\epsilon}{L}\frac{(L(\Psi(\eta)-\Psi(a))^\mu)^r}{\Gamma(r\mu+1)}(\Psi(\eta)-\Psi(a))^{1-\rho}\right)d\eta\\ &=\frac{\epsilon}{L}\frac{L^{r+1}}{\Gamma(r\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho}\mathbf{I}_{a^+}^{\mu;\,\Psi}(\Psi(t)-\Psi(a))^{r\mu}\\ &=\frac{\epsilon}{L}\frac{L^{r+1}}{\Gamma(r\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho}\frac{\Gamma(r\mu+1)}{\Gamma((r+1)\mu+1)}(\Psi(t)-\Psi(a))^{(r+1)\mu}\end{aligned}$$

Therefore,

$$\|y_{r+1}-y_r\|_{\mathbf{C}_{t;\,\Psi}}\leq\frac{\epsilon}{L}\frac{(L(\Psi(t)-\Psi(a))^\mu)^{r+1}}{\Gamma((r+1)\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho},$$

which is the inequality (4.9) for $j=r+1$. Using the principle of mathematical induction the inequality (4.9) holds for every $j\in\mathbb{N}$ and every $t\in\Delta$.

Therefore,

$$\|y_j-y_{j-1}\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}\leq\frac{\epsilon}{L}\frac{(L(\Psi(b)-\Psi(a))^\mu)^j}{\Gamma(j\mu+1)}(\Psi(b)-\Psi(a))^{1-\rho}.$$

Now using this estimation we have

$$\sum_{j=1}^\infty\|y_j-y_{j-1}\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}\leq\frac{\epsilon}{L}(\Psi(b)-\Psi(a))^{1-\rho}\sum_{j=1}^\infty\frac{(L(\Psi(b)-\Psi(a))^\mu)^j}{\Gamma(j\mu+1)}.$$

Thus we have

(4.10)
$$\sum_{j=1}^\infty\|y_j-y_{j-1}\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}\leq\frac{\epsilon}{L}(\Psi(b)-\Psi(a))^{1-\rho}\left(E_\mu(L(\Psi(b)-\Psi(a))^\mu)-1\right).$$

Hence the series

$$y_0+\sum_{j=1}^\infty(y_j-y_{j-1}) \tag{4.11}$$

converges in the weighted space $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. Let $y\in\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ such that

$$y=y_0+\sum_{j=1}^{\infty}(y_j-y_{j-1}). \tag{4.12}$$

Noting that

$$y_n=y_0+\sum_{j=1}^{n}(y_j-y_{j-1})$$

is the $n^{th}$ partial sum of the series (4.11), we have

$$\|y_n-y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}\to 0 \ as\ n\to\infty.$$

Next, we prove that this limit function $y$ is the solution of fractional integral equation with $\mathbf{I}_{a^+}^{1-\rho;\,\Psi}y^*(a)=\mathbf{I}_{a^+}^{1-\rho;\,\Psi}y(a)$. Next, by the definition of successive approximation, for any $t\in\Delta$, we have

$$\begin{aligned}
&\left|(\Psi(t)-\Psi(a))^{1-\rho}\left(y(t)-\Omega^{\rho}_{\Psi}(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}y(a)-\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)f(\eta,y(\eta))\,d\eta\right)\right|\\
&=\left|(\Psi(t)-\Psi(a))^{1-\rho}\left(y(t)-\Omega^{\rho}_{\Psi}(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}z(a)-\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,f(\eta,y(\eta))\,d\eta\right)\right|\\
&=\left|(\Psi(t)-\Psi(a))^{1-\rho}\left(y(t)-y_n(t)+\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,f(\eta,y_{n-1}(\eta))\,d\eta\right.\right.\\
&\qquad\left.\left.-\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,f(\eta,y(\eta))\,d\eta\right)\right|\\
&\leq\left|(\Psi(t)-\Psi(a))^{1-\rho}\left\{y(t)-y_n(t)\right\}\right|+\left|(\Psi(t)-\Psi(a))^{1-\rho}\,\mathbf{I}_{a^+}^{\mu;\,\Psi}\{f(t,y_{n-1}(t))-f(t,y(t))\}\right|\\
&\leq\|y-y_n\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]}+L\left[(\Psi(t)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,|y_{n-1}(\eta)-y(\eta)|\,d\eta\right]\\
&\leq\|y-y_n\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]}+L\|y_{n-1}-y\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]}\,(\Psi(t)-\Psi(a))^{1-\rho}\,\mathbf{I}_{a^+}^{\mu;\,\Psi}(\Psi(t)-\Psi(a))^{\rho-1}\\
&=\|y-y_n\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]}+\left(\frac{L\Gamma\rho}{\Gamma(\mu+\rho)}\,(\Psi(t)-\Psi(a))^{\mu}\right)\|y_{n-1}-y\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]},\ \forall n\in\mathbb{N}.
\end{aligned}$$

By taking limit as $n\to\infty$ in the above inequality, for all $t\in[a,b]$, we obtain

$$\left|(\Psi(t)-\Psi(a))^{1-\rho}\left(y(t)-\Omega^{\rho}_{\Psi}(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}y(a)-\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,f(\eta,y(\eta))\,d\eta\right)\right|=0.$$

Since, $(\Psi(t)-\Psi(a))^{1-\rho}\neq 0$ for all $t\in\Delta$, we have

$$y(t)=\Omega^{\rho}_{\Psi}(t,a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi}y(a)+\frac{1}{\Gamma(\mu)}\int_a^t\mathcal{L}^{\mu}_{\Psi}(t,\eta)\,f(\eta,y(\eta))\,d\eta,\ t\in\Delta. \tag{4.13}$$

This proves that $y$ is the solution of (1.1)-(1.2) in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$. Further, for the solution $y^*$ of inequation (4.4) and the solution $y$ of the equation (1.1), using (4.7) and (4.12), for any $t\in\Delta$, we have

$$|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t)-y(t))|$$

$$
\begin{aligned}
&= \left| (\Psi(t)-\Psi(a))^{1-\rho} \left[ y_0(t) - \left( y_0(t) + \sum_{j=1}^{\infty} (y_j(t) - y_{j-1}(t)) \right) \right] \right| \\
&\le \sum_{j=1}^{\infty} \left| (\Psi(t)-\Psi(a))^{1-\rho} (y_j(t) - y_{j-1}(t)) \right| \\
&\le \sum_{j=1}^{\infty} \| y_j - y_{j-1} \|_{\mathbf{C}_{1-\rho}[a,b]} \\
&\le \frac{\epsilon}{L} (\Psi(b)-\Psi(a))^{1-\rho} (E_\mu(L(\Psi(b)-\Psi(a))^\mu) - 1).
\end{aligned}
$$

Therefore,

$$
\|y^* - y\|_{\mathbf{C}_{1-\rho;\,\Psi}[a,b]} \le \left( \frac{(E_\mu(L(\Psi(b)-\Psi(a))^\mu) - 1)}{L} (\Psi(b)-\Psi(a))^{1-\rho} \right) \epsilon
$$

This proves the equation (1.1) is HU stable. □

**Corollary 4.6.** *Suppose that the function $f$ satisfies the assumptions of Theorem 4.5. Then, the problem* (1.1) *is generalized HU stable.*

*Proof.* Set

$$
\Psi_f(\epsilon) = \left( \frac{(E_\mu(L(\Psi(b)-\Psi(a))^\mu) - 1)}{L} (\Psi(b)-\Psi(a))^{1-\rho} \right) \epsilon,
$$

in the proof of Theorem 4.5. Then $\Psi_f(0) = 0$ and for each $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ that satisfies the inequality

$$
\left| {}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t)) \right| \le \epsilon, \ t \in \Delta,
$$

there exists a solution $y$ of equation (1.1) in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ with $\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) = \mathbf{I}_{a^+}^{1-\rho;\,\Psi} y(a)$ such that

$$
\|y^* - y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \le \Psi_f(\epsilon), \ t \in \Delta.
$$

Hence fractional differential equation (1.1) is generalized HU stable. □

**Theorem 4.7.** *Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ for any $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ and that satisfies the Lipschitz condition*

$$
|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|, \ t \in (a,b], \ y_1, y_2 \in \mathbb{R},
$$

*where $L > 0$ is a constant. For every $\epsilon > 0$, if $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ satisfies*

$$
\left| {}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t)) \right| \le \epsilon\phi(t), \quad t \in \Delta,
$$

*where* $\phi \in C(\Delta, \mathbb{R}_+)$ *is a non-decreasing function such that*

$$\left|\mathbf{I}_{a^+}^{\mu;\,\Psi}\phi(t)\right| \leq \lambda\phi(t),\ t \in \Delta$$

*and* $\lambda > 0$ *is a constant satisfying* $0 < \lambda L < 1$. *Then, there exists a solution* $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ *of equation* (1.1) *with* $\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) = \mathbf{I}_{a^+}^{1-\rho;\,\Psi} y(a)$ *such that*

$$\left|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))\right| \leq \left(\frac{\lambda}{1-\lambda L}\,(\Psi(b) - \Psi(a))^{1-\rho}\right)\epsilon\phi(t),\ t \in \Delta.$$

*Proof.* For every $\epsilon > 0$, let $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ satisfies

$$\left|{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi} y^*(t) - f(t, y^*(t))\right| \leq \epsilon\phi(t),\ t \in \Delta.$$

Proceeding as in the proof of Theorem 4.5, there exists a function $\sigma_{y^*} \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ (depending on $y^*$) such that

$$y^*(t) = \Omega_\Psi^\rho(t, a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) + \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t, y^*(t)) + \mathbf{I}_{a^+}^{\mu;\,\Psi}\sigma_{y^*}(t),\ t \in \Delta,$$

Further, using mathematical induction, one can prove that the sequence of successive approximations $\{y_n\}_{n=1}^\infty \subseteq \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ defined by

$$y_n(t) = \Omega_\Psi^\rho(t, a)\,\mathbf{I}_{a^+}^{1-\rho;\,\Psi} y^*(a) + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t, \eta)\, f(\eta, y_{n-1}(\eta))\, d\eta,\ t \in \Delta. \tag{4.14}$$

satisfy the inequality

$$\|y_j - y_{j-1}\|_{\mathbf{C}_{t;\,\Psi}} \leq \frac{\epsilon}{L}(\lambda L)^j (\Psi(t) - \Psi(a))^{1-\rho}\phi(t),\ j \in \mathbb{N}. \tag{4.15}$$

Using the inequation (4.15), we obatin

$$\sum_{j=1}^\infty \|y_j - y_{j-1}\|_{\mathbf{C}_{t;\Psi}} \leq \frac{\epsilon}{L}\left(\sum_{j=1}^\infty (\lambda L)^j\right)(\Psi(t) - \Psi(a))^{1-\rho}\,\phi(t).$$

Thus

$$\sum_{j=1}^\infty \|y_j - y_{j-1}\|_{\mathbf{C}_{t;\Psi}} \leq \epsilon\left(\frac{\lambda}{1-\lambda L}\right)(\Psi(t) - \Psi(a))^{1-\rho}\,\phi(t),\ t \in \Delta. \tag{4.16}$$

Following the steps as in the proof of the Theorem 4.5, there exists $y \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R})$ such that $\|y_n - y\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \to 0$ *as* $n \to \infty$. This $y$ is the solution of the problem (1.1)-(1.2) with $\mathbf{I}_{a^+}^{1-\rho,\Psi} y(a) = \mathbf{I}_{a^+}^{1-\rho,\Psi} y^*(a)$, and we have

$$y = y_0 + \sum_{j=1}^\infty (y_j - y_{j-1}).$$

Further, for the solution $y^*$ of inequation and the solution $y$ of the equation (1.1), for any $t \in \Delta$,

$$\begin{aligned}
&|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t)-y(t))| \\
&= \left|(\Psi(t)-\Psi(a))^{1-\rho}\left[y_0(t)-\left(y_0(t)+\sum_{j=1}^{\infty}(y_j(t)-y_{j-1}(t))\right)\right]\right| \\
&\leq \sum_{j=1}^{\infty}\left|(\Psi(t)-\Psi(a))^{1-\rho}(y_j(t)-y_{j-1}(t))\right| \\
&\leq \sum_{j=1}^{\infty}\|y_j-y_{j-1}\|_{\mathbf{C}_{t,\Psi}} \\
&= \epsilon\left(\frac{\lambda}{1-\lambda L}\right)(\Psi(t)-\Psi(a))^{1-\rho}\,\phi(t),\ t\in\Delta.
\end{aligned}$$

Thus, we have

$$|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t)-y(t))| \leq \left(\frac{\lambda}{1-\lambda L}(\Psi(b)-\Psi(a))^{1-\rho}\right)\epsilon\phi(t),\ t\in\Delta.$$

This proves the equation (1.1) is HUR stable. □

**Corollary 4.8.** *Suppose that the function $f$ satisfies the assumptions of Theorem 4.7.Then, the problem* (1.1) *is generalized HUR stable.*

*Proof.* Set $\epsilon = 1$ and $C_{f,\phi} = \left(\frac{\lambda}{1-\lambda L}(\Psi(b)-\Psi(a))^{1-\rho}\right)$ in the proof of Theorem 4.7. Then for each solution $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ that satisfies the inequality

$$\left|{}^{H}\mathbf{D}^{\mu,\nu;\,\Psi}_{a^+}y^*(t)-f(t,y^*(t))\right| \leq \phi(t),\ t\in\Delta,$$

there exists a solution $y$ of equation (1.1) in $\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ with $\mathbf{I}^{1-\rho;\,\Psi}_{a^+}y^*(a) = \mathbf{I}^{1-\rho;\,\Psi}_{a^+}y(a)$ such that

$$|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t)-y(t))| \leq C_{f,\phi}\,\phi(t),\ t\in\Delta.$$

Hence the fractional differential equation (1.1) is generalized HUR stable. □

## 5. $\epsilon-$Approximate Solutions to Hilfer FDE

**Definition 5.1.** A function $y^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})$ that satisfies the fractional differential inequality

$$\left|{}^{H}\mathbf{D}^{\mu,\nu;\,\Psi}_{a^+}y^*(t)-f(t,y^*(t))\right| \leq \epsilon,\quad t\in\Delta,$$

is called an *$\epsilon$-approximate solution* of Ψ–Hilfer FDE (1.1).

**Theorem 5.2.**([15]) *Let $\mathfrak{u}$, $\mathbf{v}$ be two integrable, non negative functions and $\mathbf{g}$ be a continuous, nonnegative, nondecreasing function with domain $\Delta$. If*

$$\mathfrak{u}(t) \leq \mathbf{v}(t) + \mathbf{g}(t) \int_a^t \mathcal{L}_\Psi^\mu(\tau, s)\mathfrak{u}(\tau)\, d\tau,$$

*then*

$$\mathfrak{u}(t) \leq \mathbf{v}(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[\mathbf{g}(t)\,\Gamma(\mu)]^k}{\Gamma(\mu k)} \mathcal{L}_\Psi^{\mu k}(t, \tau)\mathbf{v}(\tau)\, d\tau, \forall\, t \in \Delta. \tag{5.1}$$

**Theorem 5.3.** *Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies Lipschitz condition*

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

*for each $t \in (a, b]$ and all $y_1, y_2 \in \mathbb{R}$, where $L > 0$ is constant. Let ${y_i}^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}), (i = 1, 2)$ be an $\epsilon_i$-approximte solutions of FDE (1.1) corresponding to $\mathbf{I}_{a^+}^{1-\rho;\,\Psi}{y_i}^*(a) = y_a^{(i)} \in \mathbb{R}$, respectively. Then,*

$$\begin{aligned}
&\|{y_1}^* - {y_2}^*\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})}\\
&\leq (\epsilon_1 + \epsilon_2)\left(\frac{(\Psi(b) - \Psi(a))^{\mu-\rho+1}}{\Gamma(\mu + 1)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma((k + 1)\mu - \rho + 1)}(\Psi(b) - \Psi(a))^{(k+1)\mu}\right)\\
&\quad + |y_a^{(1)} - y_a^{(2)}| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma(\rho + k\mu)}(\Psi(b) - \Psi(a))^{k\mu}\right).
\end{aligned} \tag{5.2}$$

*Proof.* Let ${y_i}^* \in \mathbf{C}_{1-\rho;\,\Psi}(\Delta, \mathbb{R}), (i = 1, 2)$ be an $\epsilon_i$-approximate solution of FDE (1.1) that satisfies the initial condition $\mathbf{I}_{a^+}^{1-\rho;\,\Psi}{y_i}^*(a) = y_a^{(i)} \in \mathbb{R}$. Then,

$$\left|{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi}{y_i}^*(t) - f(t, {y_i}^*(t))\right| \leq \epsilon_i,\ t \in \Delta. \tag{5.3}$$

Operating $\mathbf{I}_{a^+}^{\mu;\,\Psi}$ on both the sides of the above inequation and using the Lemma 2.5, we get

$$\begin{aligned}
\mathbf{I}_{a^+}^{\mu;\,\Psi}\epsilon_i &\geq \mathbf{I}_{a^+}^{\mu;\,\Psi}\left|{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi}{y_i}^*(t) - f(t, {y_i}^*(t))\right|\\
&\geq \left|\frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_\Psi^\mu(t, \eta)\left({}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi}{y_i}^*(\eta) - f(t, {y_i}^*(\eta))\right) d\eta\right|\\
&= \left|\mathbf{I}_{a^+}^{\mu;\,\Psi}\,{}^H\mathbf{D}_{a^+}^{\mu,\nu;\,\Psi}{y_i}^*(t) - \mathbf{I}_{a^+}^{\mu;\,\Psi}f(t, {y_i}^*(t))\right|\\
&= \left|{y_i}^*(t) - \mathbf{I}_{a_+}^{1-\rho;\Psi}{y_i}^*(a)\,\Omega_\Psi^\rho(t, a) - \mathbf{I}_{a^+}^{\mu;\,\Psi}f(t, {y_i}^*(t))\right|.
\end{aligned}$$

Therefore,

$$\frac{\epsilon_i}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} \geq \left| {y_i}^*(t) - y_a^{(i)}\,\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,{y_i}^*(t))\right|,\ i=1,2. \tag{5.4}$$

Using the following inequalities

$$|x-y| \leq |x|+|y| \text{ and } |x|-|y| \leq |x-y|,\ x,y\in\mathbb{R},$$

from the inequation (5.4), for any $t\in\Delta$, we have

$$\begin{aligned}
&\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu}\\
&\geq \left|{y_1}^*(t) - y_a^{(1)}\,\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,{y_1}^*(t))\right|\\
&+ \left|{y_2}^*(t) - y_a^{(2)}\,\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,{y_2}^*(t))\right|\\
&\geq \left|\left({y_1}^*(t) - y_a^{(1)}\,\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,{y_1}^*(t))\right)\right.\\
&\qquad \left.-\left({y_2}^*(t) - y_a^{(2)}\,\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a^+}^{\mu;\,\Psi} f(t,{y_2}^*(t))\right)\right|\\
&= \left|({y_1}^*(t)-{y_2}^*(t)) - (y_a^{(1)}-y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a) - \mathbf{I}_{a+}^{\mu}[f(t,{y_1}^*(t)) - f(t,{y_2}^*(t))]\right|\\
&\geq |({y_1}^*(t)-{y_2}^*(t))| - \left|(y_a^{(1)}-y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right|\\
&\qquad - \left|\mathbf{I}_{a+}^{\mu}\ \{f(t,{y_1}^*(t)) - f(t,{y_2}^*(t))\}\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|({y_1}^*(t)-{y_2}^*(t))| &\leq \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)}-y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right|\\
&\quad + \left|\mathbf{I}_{a^+}^{\mu;\,\Psi}(f(t,{y_1}^*(t)) - f(t,{y_2}^*(t)))\right|\\
&\leq \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)}-y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right|\\
&+ \frac{L}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)\,|{y_1}^*(\eta)) - {y_2}^*(\eta)|\,d\eta.
\end{aligned}$$

Applying Theorem 5.2. with

$$\begin{aligned}
\mathfrak{u}(t) &= |{y_1}^*(t)-{y_2}^*(t)|,\\
\mathbf{v}(t) &= \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)}-y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right|,\\
\mathbf{g}(t) &= \frac{L}{\Gamma(\mu)},
\end{aligned}$$

we obtain

$$
\begin{aligned}
&|{y_1}^*(t) - {y_2}^*(t)| \\
&\leq \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_\Psi^{\rho}(t,a)\right| \\
&\quad + \int_a^t \sum_{k=1}^{\infty} \frac{L^k}{\Gamma(k\mu)} \mathcal{L}_\Psi^{k\mu}(t,\eta) \left( \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(\eta)-\Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_\Psi^{\rho}(t,a)\right| \right) d\eta \\
&= \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_\Psi^{\rho}(t,a)\right| \\
&\quad + \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)} \sum_{k=1}^{\infty} L^k\, \mathbf{I}_{a^+}^{k\mu;\,\Psi} (\Psi(t)-\Psi(a))^{\mu} \\
&\quad + \frac{|y_a^{(1)} - y_a^{(2)}|}{\Gamma(\rho)} \sum_{k=1}^{\infty} L^k\, \mathbf{I}_{a^+}^{k\mu;\,\Psi} (\Psi(t)-\Psi(a))^{\rho-1} \\
&= \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)}\,(\Psi(t)-\Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_\Psi^{\rho}(t,a)\right| \\
&\quad + \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\mu+1)} \sum_{k=1}^{\infty} L^k\, \frac{\Gamma(\mu+1)}{\Gamma((k+1)\mu+1)} (\Psi(t)-\Psi(a))^{(k+1)\mu} \\
&\quad + \frac{|y_a^{(1)} - y_a^{(2)}|}{\Gamma(\rho)} \sum_{k=1}^{\infty} \frac{L^k\,\Gamma(\rho)}{\Gamma(\rho+k\mu)} (\Psi(t)-\Psi(a))^{k\mu+\rho-1} \\
&= (\epsilon_1+\epsilon_2) \left( \frac{(\Psi(t)-\Psi(a))^{\mu}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma((k+1)\mu+1)} (\Psi(t)-\Psi(a))^{(k+1)\mu} \right) \\
&+ |y_a^{(1)} - y_a^{(2)}| \left( \frac{(\Psi(t)-\Psi(a))^{\rho-1}}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma(\rho+k\mu)} (\Psi(t)-\Psi(a))^{k\mu+\rho-1} \right).
\end{aligned}
$$

Thus for every $t \in \Delta$, we have

$$
\begin{aligned}
&(\Psi(t)-\Psi(a))^{1-\rho} |({y_1}^*(t) - {y_2}^*(t))| \\
&\leq (\epsilon_1+\epsilon_2) \left( \frac{(\Psi(t)-\Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma((k+1)\mu-\rho+1)} (\Psi(t)-\Psi(a))^{(k+1)\mu} \right) \\
&+ |y_a^{(1)} - y_a^{(2)}| \left( \frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma(\rho+k\mu)} (\Psi(t)-\Psi(a))^{k\mu} \right) \\
&\leq (\epsilon_1+\epsilon_2) \left( \frac{(\Psi(b)-\Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma((k+1)\mu-\rho+1)} (\Psi(b)-\Psi(a))^{(k+1)\mu} \right) \\
&+ |y_a^{(1)} - y_a^{(2)}| \left( \frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{L^k}{\Gamma(\rho+k\mu)} (\Psi(b)-\Psi(a))^{k\mu} \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|{y_1}^* - {y_2}^*\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \\
&\leq (\epsilon_1+\epsilon_2)\left(\frac{(\Psi(b)-\Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty}\frac{L^k}{\Gamma((k+1)\mu-\rho+1)}(\Psi(b)-\Psi(a))^{(k+1)\mu}\right) \\
&+ |y_a^{(1)} - y_a^{(2)}|\left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty}\frac{L^k}{\Gamma(\rho+k\mu)}(\Psi(b)-\Psi(a))^{k\mu}\right),
\end{aligned}
$$

which is the desired inequality. □

**Remark 5.4.** If $\epsilon_1 = \epsilon_2 = 0$ in the inequality (5.3) then ${y_1}^*$ and ${y_2}^*$ are the solutions of Cauchy problem (1.1)–(1.2) in the space $\mathbf{C}_{1-\rho;\,\Psi}[a,b]$. Further, for $\epsilon_1 = \epsilon_2 = 0$ the inequality takes the form

$$
\|{y_1}^* - {y_2}^*\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} \leq |y_a^{(1)} - y_a^{(2)}|\left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty}\frac{L^k}{\Gamma(\rho+k\mu)}(\Psi(b)-\Psi(a))^{k\mu}\right),
$$

which provides the information regarding the continuous dependence of the solution of the problem (1.1)-(1.2) on the initial condition. In addition, if $y_a^{(1)} = y_a^{(2)}$ we have $\|{y_1}^* - {y_2}^*\|_{\mathbf{C}_{1-\rho;\,\Psi}(\Delta,\mathbb{R})} = 0$, which gives the uniqueness of solution of the problem (1.1)–(1.2).

## 6. Examples

**Example 6.1.** Consider the Ψ–Hilfer FDEs

$$
{}^{H}\mathbf{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\,\Psi} y(t) = 4y(t),\ t \in J = [0,1], \tag{6.1}
$$

$$
\mathbf{I}_{0^+}^{\frac{1}{4}} y(0) = 2. \tag{6.2}
$$

comparing with the Cauchy problem (1.1)-(1.2), we have

$$
\mu = \frac{1}{2},\ \nu = \frac{1}{2},\ \rho = \mu+\nu-\mu\nu = \frac{3}{4},\ y_0^* = \mathbf{I}_{0^+}^{1-\frac{3}{4};\,\Psi} y^*(0) = 2, \text{ and } f(t,y(t)) = 4y(t).
$$

Clearly, $f$ satisfies Lipschitz condition with Lipschitz constant $L = 4$. By Theorem 3.2. the initial value problem (6.1)-(6.2) has a unique solution. Further, the Theorem 4.5. guarantees that the equation (6.1) is HU stable. Indeed, we prove that for given $\epsilon > 0$ and solution $y^*$ of the inequality

$$
\left|{}^{H}\mathbf{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\,\Psi} y^*(t) - 4y^*(t)\right| \leq \epsilon,\ t \in [0,1],
$$

we can find a constant $C$ and a solution $y$ of the given equation (6.1) such that

$$
\|y^* - y\|_{\mathbf{C}_{1-\frac{3}{4};\,\Psi}} \leq C\,\epsilon.
$$

For example, take $\epsilon = 8$ and consider the inequality

$$\left| {}^{H}\mathbf{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\Psi} y^*(t) - 4y^*(t) \right| \leq 8, \ t \in [0,1]. \tag{6.3}$$

Note that the function $y^*(t) = 2\frac{(\Psi(t)-\Psi(0))^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})}$ satisfies the inequality (6.3). Further,

$${}^{H}\mathbf{D}_{0^+}^{\frac{1}{2},\frac{1}{2};\Psi} y^*(t) = 0,$$

which shows $y^*$ is not the solution of the Cauchy problem (6.1)-(6.2). Next, as discussed in the proof of Theorem 4.5, we define the sequence of successive approximations to the solution of (6.1) as follows:

$$\begin{aligned}
y_0(t) &= y^*(t) = 2\frac{(\Psi(t)-\Psi(0))^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} \\
y_n(t) &= \Omega_{\Psi}^{\frac{3}{4}}(t,a)\, \mathbf{I}_{a^+}^{1-\frac{3}{4};\Psi} y^*(a) + \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)\, f(\eta, y_{n-1}(\eta))\, d\eta \\
&= \frac{(\Psi(t)-\Psi(0))^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} + \frac{4}{\Gamma(\frac{1}{2})} \int_0^t \mathcal{L}_{\Psi}^{\frac{1}{2}}(t,\eta)\, y_{n-1}(\eta))\, d\eta, \ t \in J, n \in \mathbb{N}.
\end{aligned}$$

Then,

$$\begin{aligned}
y_1(t) &= 2\frac{(\Psi(t)-\Psi(0))^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} + 8\frac{(\Psi(t)-\Psi(0))^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \\
y_2(t) &= 2(\Psi(t)-\Psi(0))^{-\frac{1}{4}} \left[ \frac{1}{\Gamma(\frac{3}{4})} + 4\frac{(\Psi(t)-\Psi(0))^{\frac{1}{2}}}{\Gamma(\frac{5}{4})} + \frac{16}{\Gamma(\frac{7}{4})} (\Psi(t)-\Psi(0)) \right].
\end{aligned}$$

In general, we have

$$y_n(t) = 2(\Psi(t)-\Psi(0))^{-\frac{1}{4}} \sum_{j=0}^{n} \frac{(4(\Psi(t)-\Psi(0))^{\frac{1}{2}})^j}{\Gamma(j\frac{1}{2}+\frac{3}{4})}, n \in \mathbb{N}.$$

The exact solution of the initial value problem (6.1)-(6.2) is given by

$$\begin{aligned}
y(t) &= \lim_{n\to\infty} y_n(t) \\
&= \lim_{n\to\infty} 2(\Psi(t)-\Psi(0))^{-\frac{1}{4}} \sum_{j=0}^{n} \frac{(4(\Psi(t)-\Psi(0))^{\frac{1}{2}})^j}{\Gamma(j\frac{1}{2}+\frac{3}{4})} \\
&= 2(\Psi(t)-\Psi(0))^{-\frac{1}{4}}\, E_{\frac{1}{2},\frac{3}{4}}(4(\Psi(t)-\Psi(0))^{\frac{1}{2}}). \qquad (6.4)
\end{aligned}$$

Therefore

$$\|y^* - y\|_{\mathbf{C}_{1-\frac{3}{4};\Psi}} = \max_{t\in[0,1]} \left| (\Psi(t)-\Psi(0))^{1-\frac{3}{4}} (y^*(t) - y(t)) \right|$$

$$
\begin{aligned}
&= \max_{t\in[0,1]} \left| 2E_{\frac{1}{2},\frac{3}{4}}(4(\Psi(t)-\Psi(0))^{\frac{1}{2}}) - \frac{2}{\Gamma(\frac{3}{4})} \right| \\
&\leq \left| 2E_{\frac{1}{2},\frac{3}{4}}(4(\Psi(1)-\Psi(0))^{\frac{1}{2}}) - \frac{2}{\Gamma(\frac{3}{4})} \right| \\
&= C_f\,\epsilon,
\end{aligned}
$$

where $C_f = \frac{1}{8}\left| 2E_{\frac{1}{2},\frac{3}{4}}(4(\Psi(1)-\Psi(0))^{\frac{1}{2}}) - \frac{2}{\Gamma(\frac{3}{4})} \right|$. Along a similar line, for each $\epsilon > 0$ and for each solution $y^* \in \mathbf{C}_{1-\frac{3}{4};\,\Psi}[a,b]$ of the inequation (4.1), one can find by the method of successive approximation a solution $y \in \mathbf{C}_{1-\frac{3}{4};\,\Psi}[a,b]$ of (6.1) that satisfies the inequality

$$\|y^* - y\|_{\mathbf{C}_{1-\frac{3}{4};\,\Psi}[a,b]} \leq C_f\,\epsilon.$$

**Example 6.2.** Consider the nonlinear Ψ–Hilfer FDEs

$$
\begin{aligned}
&(6.5) \qquad {}^H\mathbf{D}^{\mu,\nu;\,\Psi}_{0+} y(t) = \frac{3\sqrt{\pi}}{4}\,(\Psi(t)-\Psi(0)) + (\Psi(t)-\Psi(0))^3 - y^2(t),\ t\in[0,1],\\
&(6.6) \qquad \mathbf{I}^{1-\rho}_{0+} y(0) = 0.
\end{aligned}
$$

Define $f : [0,1]\times[-b,b] \to \mathbb{R}$, $0 < b < \infty$ by

$$f(t,y) = \frac{3\sqrt{\pi}}{4}\,(\Psi(t)-\Psi(0)) + (\Psi(t)-\Psi(0))^3 - y^2.$$

Then, for any $t \in [0,1]$ and $u, v \in [-b,b]$, we have

$$|f(t,u) - f(t,v)| = |u^2 - v^2| \leq 2b|u-v|.$$

Since $f$ satisfies the Lipschitz condition with $L = 2b$, by Theorem 3.2, problem (6.5)-(6.6) has a unique solution and by Theorem 4.5, (6.5) is HU stable.

In particular, for $\mu = \frac{1}{2}, \nu = 1$ and $\Psi(t) = t$, the problem (6.5)-(6.6) reduces to the following nonlinear Caputo FDEs

$$
\begin{aligned}
&(6.7) \qquad {}^C\mathbf{D}^{\frac{1}{2}}_{0+} y(t) = \frac{3\sqrt{\pi}}{4}\,t + t^3 - y^2(t),\ t\in[0,1],\\
&(6.8) \qquad y(0) = 0.
\end{aligned}
$$

Note that for $\epsilon = 2.5$ and $y^*(t) = 0, t \in [0,1]$, we have $|{}^C\mathbf{D}^{\frac{1}{2}}_{0+} y^*(t) - f(t, y^*(t))| < \epsilon, t \in [0,1]$. Further, $y^*(t)$ is not solution of the problem (6.7)-(6.8). Therefore, for the Caputo FDEs (6.7)-(6.8) the successive approximation defined by (4.7)-(4.8) takes the form

$$
\begin{aligned}
&y_0(t) = y^*(t) = 0,\ t\in[0,1]\\
&y_n(t) = y^*(0) + \mathbf{I}^{\frac{1}{2}}_{0+} f(t, y_{n-1})(t) = \mathbf{I}^{\frac{1}{2}}_{0+} f(t, y_{n-1})(t),\ t\in[0,1],\ n\geq 1.
\end{aligned}
$$

One can verify that the first four successive approximations $y_0, y_1, y_2, y_3$ to the solution of (6.7)-(6.8) are

$$y_0(t) = y^*(t) = 0,$$
$$y_1(t) = t^{\frac{3}{2}} + (0.5158)t^{\frac{7}{2}},$$
$$y_2(t) = t^{\frac{3}{2}} + (0.0955)t^{\frac{15}{2}} - (0.1041)t^{\frac{11}{2}},$$
$$y_3(t) = t^{\frac{3}{2}} - (0.0023)t^{\frac{31}{2}} - (0.0032)t^{\frac{23}{2}} - (0.0612)t^{\frac{19}{2}} + (0.07477)t^{\frac{15}{2}} + (0.0054)t^{\frac{27}{2}}.$$

Further, by direct substitution one can easily check that $y(t) = t^{\frac{3}{2}}$, $t \in [0,1]$ is the exact solution of the problem (6.7)-(6.8).

From Fig.1, it very well may be seen that the successive approximations given above are converging with the exact solution.

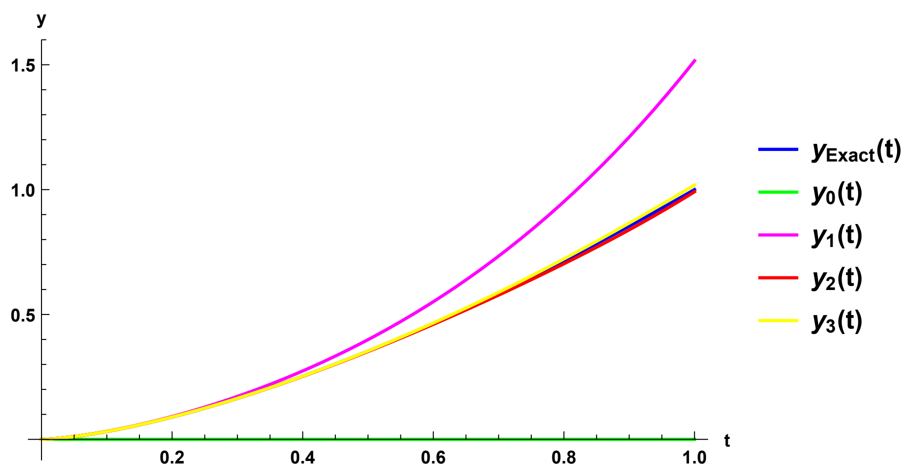


Figure: 1

Next, we examine the HU stability of equation (6.7), by demonstrating that for each $\epsilon > 0$ and each solution $y^* \in C([0,1], \mathbb{R})$ of the inequality

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y^*(t) - f(t, y^*(t)) \right| < \epsilon, \text{ where } f(t, y^*(t)) = \frac{3\sqrt{\pi}}{4}\, t + t^3 - {y^*}^2(t),\ t \in [0,1],$$

we have a solution $y(t) = t^{\frac{3}{2}},\ t \in [0,1]$ of the problem (6.7)-(6.8) such that

$$\|y - y^*\| < C_f\, \epsilon, \text{ for some } C_f > 0.$$

Indeed,

(i) For $y_1^*(t) = t^{\frac{6}{5}}$ and $\epsilon = 3.7$, we have

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y_1^*(t) - f(t, y_1^*(t)) \right| < \epsilon, \quad \text{and } \|y_1^* - y\| < C_f\, \epsilon,\ C_f = 0.55.$$

(ii) For $y_2^*(t) = t$ and $\epsilon = 4.5$, we have

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y_2^*(t) - f(t, y_2^*(t)) \right| < \epsilon, \quad \text{and } \|y_2^* - y\| < C_f\, \epsilon,\ C_f = 0.44.$$

(iii) For $y_3^*(t) = t^{\frac{7}{5}}$ and $\epsilon = 4.7$, we have

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y_3^*(t) - f(t, y_3^*(t)) \right| < \epsilon, \quad \text{and } \|y_3^* - y\| < C_f\, \epsilon,\ C_f = 0.43.$$

(iv) For $y_4^*(t) = t^2$ and $\epsilon = 4.9$, we have

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y_4^*(t) - f(t, y_4^*(t)) \right| < \epsilon, \quad \text{and } \|y_4^* - y\| < C_f\, \epsilon,\ C_f = 0.41.$$

(v) For $y_5^*(t) = t^{\frac{5}{2}}$ and $\epsilon = 5$, we have

$$\left| {}^C\mathbf{D}_{0+}^{\frac{1}{2}} y_5^*(t) - f(t, y_5^*(t)) \right| < \epsilon, \quad \text{and } \|y_5^* - y\| < C_f\, \epsilon,\ C_f = 0.40.$$

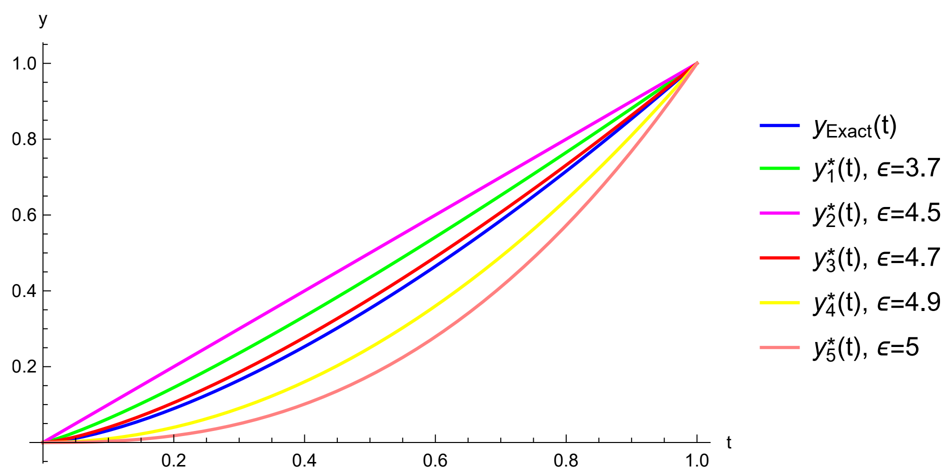


Figure: 2

**Remark 6.3.** From Fig.2, it follows that the $\epsilon$–approximated solutions $y_i^*(t), (i = 1, 2, 3, 4, 5)$ approaches to the exact solution $y_{Exact}(t)$ when $\epsilon \to 0$.

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