J. Korean Math. Soc. **57** (2020), No. 6, pp. 1335–1345
https://doi.org/10.4134/JKMS.j190640
pISSN: 0304-9914 / eISSN: 2234-3008

A CERTAIN KÄHLER POTENTIAL OF THE POINCARÉ METRIC AND ITS CHARACTERIZATION

Young-Jun Choi, Kang-Hyurk Lee, and Sungmin Yoo

Abstract. We will show a rigidity of a Kähler potential of the Poincaré metric with a constant length differential.

1. Introduction

From the fundamental result of Donnelly-Fefferman [4], the vanishing of the space of L^2 harmonic (p,q) forms has been an important research theme in the theory of complex domains. Since M. Gromov ([6], see also [2]) suggested the concept of the Kähler hyperbolicity and gave a connection to the vanishing theorem, there have been many studies on the Kähler hyperbolicity of the Bergman metric, which is a fundamental Kähler structure of bounded pseudoconvex domains. The Kähler structure ω is *Kähler hyperbolic* if there is a global 1-form η with $d\eta = \omega$ and $\sup \|\eta\|_\omega < \infty$.

In [3], H. Donnelly showed the Kähler hyperbolicity of Bergman metric on some class of weakly pseudoconvex domains. For bounded homogeneous domain D in $\mathbb{C}^n$ and its Bergman metric ω_D especially, he used a classical result of Gindikin [5] to show that $\sup \|d \log K_D\|_{\omega_D} < \infty$. Here K_D is the Bergman kernel function of D so $\log K_D$ is a canonical potential of ω_D.

In their paper [7], S. Kai and T. Ohsawa gave another approach. They proved that every bounded homogeneous domain has a Kähler potential of the Bergman metric whose differential has a constant length.

Theorem 1.1 (Kai-Ohsawa [7]). *For a bounded homogeneous domain D in $\mathbb{C}^n$, there exists a positive real valued function φ on D such that $\log \varphi$ is a Kähler potential of the Bergman metric ω_D and $\|d \log \varphi\|_{\omega_D}$ is constant.*

Received September 17, 2019; Accepted December 4, 2019.
2010 *Mathematics Subject Classification.* 32Q15, 32M25, 30F45.
Key words and phrases. The Poincaré metric, Kähler hyperbolicity.
The research of first and second named authors was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (No. 2018R1C1B3005963, No. NRF-2019R1F1A1060891). The last named author was supported by IBS-R003-D1, Institute for Basic Science in Korea.

It can be obtained by the facts that each homogeneous domain is biholomorphic to a Siegel domain (see [10]) and a homogeneous Siegel domain is affine homogeneous (see [8]).

More precisely, let us consider a bounded homogeneous domain D in $\mathbb{C}^n$ and a biholomorphism $F : D \to S$ for a Siegel domain S. For the Bergman kernel function K_S of S which is a canonical potential of the Bergman metric ω_S, it is easy to show that $d \log K_S$ has a constant length with respect to ω_S from the affine homogeneity of S (the group of affine holomorphic automorphisms acts transitively on S). Since $\log K_S$ is a Kähler potential of ω_S, the transformation formula of the Bergman kernel implies that the pullback $F^* \log K_S = \log K_S \circ F$ is also a Kähler potential of ω_D. Using the fact that $F : (D, \omega_D) \to (S, \omega_S)$ is an isometry, we have $\|d(F^* \log K_S)\|_{\omega_D} = \|d \log K_S\|_{\omega_S} \circ F$. As a function φ in Theorem 1.1, we can choose the pullback $K_S \circ F$ of the Bergman kernel function of the Siegel domain.

At this junction, it is natural to ask:

> *If there is a Kähler potential* $\log \varphi$ *with a constant* $\|d \log \varphi\|_{\omega_D}$*, is it always obtained by the pullback of the Bergman kernel function of the Siegel domain?*

The aim of this paper is to discuss of this question in the 1-dimensional case.

The only bounded homogeneous domain in $\mathbb{C}$ is the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ up to the biholomorphic equivalence and the 1-dimensional correspondence of the Bergman metric, namely a holomorphically invariant hermitian structure, is only the Poincaré metric. Hence the main theorem as follows gives a positive answer to the question.

Theorem 1.2. *Let* ω_Δ *be the Poincaré metric of the unit disc* Δ*. Suppose that there exists a positive real valued function* $\varphi : \Delta \to \mathbb{R}$ *such that* $\log \varphi$ *is a Kähler potential of the Poincaré metric and* $\|d \log \varphi\|_{\omega_\Delta}$ *is constant on* Δ*. Then* φ *is the pullback of the canonical potential on the half-plane* $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

Note that 1-dimensional Siegel domain is just the half-plane. We will introduce the Poincaré metric and related notions in Section 2. As an application of the main theorem, we can characterize the half-plane by the canonical potential.

Corollary 1.3. *Let* D *be a simply connected, proper domain in* $\mathbb{C}$ *with a Poincaré metric* $\omega_D = i\lambda dz \wedge d\bar{z}$*. If* $\|d \log \lambda\|_{\omega_D}$ *is constant on* D*, then* D *is affine equivalent to the half-plane* $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

In Section 2, we will introduce notions and concrete version of the main theorem. Then we will study the existence of a nowhere vanishing complete holomorphic vector field which is tangent to a potential whose differential is of constant length (Section 3). Using relations between complete holomorphic vector fields and model potentials in Section 4, we will prove theorems.

2. Background materials

Let X be a Riemann surface. The Poincaré metric of X is a complete hermitian metric with a constant Gaussian curvature, -4. The Poincaré metric exists on X if and only if X is a quotient of the unit disc. If X is covered by Δ, the Poincaré metric can be induced by the covering map $\pi : \Delta \to X$ and it is uniquely determined. Throughout of this paper, the Kähler form of the Poincaré metric of X, denoted by ω_X, stands for the metric also. When $\omega_X = i\lambda dz \wedge d\bar{z}$ in the local holomorphic coordinate function z, the curvature can be written by

$$\kappa = -\frac{2}{\lambda}\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda.$$

So the curvature condition $\kappa \equiv -4$ implies that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda = 2\lambda,$$

equivalently

$$dd^c \log \lambda = 2\omega_X,$$

where $d^c = \frac{i}{2}(\overline{\partial} - \partial)$. That means the function $\frac{1}{2}\log\lambda$ is a local Kähler potential of ω_X. Any other local potential of ω_X is always of the form $\frac{1}{2}\log\lambda + \log|f|^2$ where f is a local holomorphic function on the domain of z. We call $\frac{1}{2}\log\lambda$ the *canonical potential* with respect to the coordinate function z. For a domain D in $\mathbb{C}$, the canonical potential of D means the canonical potential with respect to the standard coordinate function of $\mathbb{C}$.

Let us consider the Poincaré metric ω_Δ of the unit disc Δ:

$$\omega_\Delta = i\frac{1}{\left(1-|z|^2\right)^2} dz \wedge d\bar{z} = i\lambda_\Delta dz \wedge d\bar{z}.$$

The canonical potential λ_Δ satisfies

$$\begin{aligned} \|d \log \lambda_\Delta\|^2_{\omega_\Delta} &= \left\| \frac{\partial \log \lambda_\Delta}{\partial z} dz + \frac{\partial \log \lambda_\Delta}{\partial \bar{z}} d\bar{z} \right\|^2_{\omega_\Delta} \\ &= \frac{\partial \log \lambda_\Delta}{\partial z}\frac{\partial \log \lambda_\Delta}{\partial \bar{z}}\frac{1}{\lambda_\Delta} = 4|z|^2, \end{aligned}$$

so does not have a constant length. By the same way of Kai-Ohsawa [7], we can get a model for φ in Theorem 1.1 for the unit disc,

$$\varphi_\theta(z) = \frac{\left|1+e^{i\theta}z\right|^4}{\left(1-|z|^2\right)^2} \quad \text{for } \theta \in \mathbb{R} \tag{2.1}$$

as a pullback of the canonical potential $\lambda_{\mathbf{H}} = 1/|\mathrm{Re}\, w|^2$ on the left-half plane $\mathbf{H} = \{w : \mathrm{Re}\, w < 0\}$ by the Cayley transforms (see (4.3) for instance). The term θ depends on the choice of the Cayley transform. Since $\log\varphi_\theta = \log\lambda_\Delta + \log\left|1+e^{i\theta}z\right|^4$, the function $\frac{1}{2}\log\varphi_\theta$ is a Kähler potential. Moreover

$$\|d\log\varphi_\theta\|^2_{\omega_\Delta} \equiv 4.$$

At this moment, we introduce a significant result of Kai-Ohsawa.

Theorem 2.1 (Kai-Ohsawa [7]). *For a bounded homogeneous domain D in $\mathbb{C}^n$, suppose that there is a Kähler potential $\log\psi$ of the Bergman metric ω_D with a constant $\|d\log\psi\|_{\omega_D}$, then $\|d\log\psi\|_{\omega_D} = \|d\log\varphi\|_{\omega_D}$ where φ is as in Theorem 1.1.*

Suppose that a positively real valued φ on Δ satisfies that $dd^c\log\varphi = 2\omega_\Delta$ and $\|d\log\varphi\|^2_{\omega_\Delta} \equiv c$ for some constant c. Theorem 2.1 implies that c must be 4. Therefore, we can rewrite Theorem 1.2 by:

Theorem 2.2. *If there exists a function $\varphi : \Delta \to \mathbb{R}$ satisfying*

$$dd^c\log\varphi = 2\omega_\Delta \quad \text{and} \quad \|d\log\varphi\|^2_{\omega_\Delta} \equiv 4, \tag{2.2}$$

then $\varphi = r\varphi_\theta$ as in (2.1) *for some $r > 0$ and $\theta \in \mathbb{R}$.*

Corollary 1.3 can be also written by:

Corollary 2.3. *Let D be a simply connected, proper domain in $\mathbb{C}$ with a Poincaré metric $\omega_D = i\lambda dz\wedge d\bar{z}$. If $\|d\log\lambda\|^2_{\omega_D} \equiv 4$, then D is affine equivalent to the half-plane $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.*

3. Existence of nowhere vanishing complete holomorphic vector field

In this section, we will study an existence of a complete holomorphic tangent vector field on a Riemann surface X which admits a Kähler potential of the Poincaré metric with a constant length differential.

By a holomorphic tangent vector field of a Riemann surface X, we means a holomorphic section $\mathcal{W}$ to the holomorphic tangent bundle $T^{1,0}X$. If the corresponding real tangent vector field $\operatorname{Re}\mathcal{W} = \mathcal{W} + \overline{\mathcal{W}}$ is complete, we also say $\mathcal{W}$ is complete. Thus the complete holomorphic tangent vector field generates a 1-parameter family of holomorphic transformations.

In this section, we will show that:

Theorem 3.1. *Let X be a Riemann surface with the Poincaré metric ω_X. If there is a function $\varphi : X \to \mathbb{R}$ with*

$$dd^c\log\varphi = 2\omega_X \quad \text{and} \quad \|d\log\varphi\|^2_{\omega_X} \equiv 4, \tag{3.1}$$

then there is a nowhere vanishing complete holomorphic vector field $\mathcal{W}$ such that $(\operatorname{Re}\mathcal{W})\varphi \equiv 0$.

Proof. Take a local holomorphic coordinate function z and let $\omega_X = i\lambda dz \wedge d\bar{z}$. The equation (3.1) can be written by

$$(\log\varphi)_{z\bar{z}} = 2\lambda \quad \text{and} \quad (\log\varphi)_z\,(\log\varphi)_{\bar{z}} = 4\lambda.$$

Here, $(\log\varphi)_z = \frac{\partial}{\partial z}\log\varphi$, $(\log\varphi)_{\bar{z}} = \frac{\partial}{\partial \bar{z}}\log\varphi$ and $(\log\varphi)_{z\bar{z}} = \frac{\partial^2}{\partial z\partial\bar{z}}\log\varphi$. This implies that

$$\begin{aligned}
\left(\varphi^{-1/2}\right)_z &= \frac{\partial}{\partial z}\varphi^{-1/2} = -\frac{1}{2}\varphi^{-1/2}\,(\log\varphi)_z\,;\\
\left(\varphi^{-1/2}\right)_{z\bar{z}} &= \frac{\partial^2}{\partial z\partial\bar{z}}\varphi^{-1/2} = -\frac{1}{2}\varphi^{-1/2}\,(\log\varphi)_{z\bar{z}} + \frac{1}{4}\varphi^{-1/2}\,(\log\varphi)_z\,(\log\varphi)_{\bar{z}}\\
&= -\frac{1}{2}\varphi^{-1/2}\left((\log\varphi)_{z\bar{z}} - \frac{1}{2}(\log\varphi)_z\,(\log\varphi)_{\bar{z}}\right)\\
&= 0.
\end{aligned}$$

Thus we have that the function $\varphi^{-1/2}$ is harmonic so $\left(\varphi^{-1/2}\right)_z$ is holomorphic.

Let us consider a local holomorphic vector field,

$$\mathcal{W} = \frac{i}{\left(\varphi^{-1/2}\right)_z}\frac{\partial}{\partial z} = \frac{-2i\varphi^{3/2}}{\varphi_z}\frac{\partial}{\partial z} = \frac{-2i\varphi^{1/2}}{(\log\varphi)_z}\frac{\partial}{\partial z}.$$

In any other local holomorphic coordinate function w, we have

$$\mathcal{W} = \frac{i}{\left(\varphi^{-1/2}\right)_z}\frac{\partial}{\partial z} = \frac{i}{\left(\varphi^{-1/2}\right)_w \frac{\partial w}{\partial z}}\frac{\partial w}{\partial z}\frac{\partial}{\partial w} = \frac{i}{\left(\varphi^{-1/2}\right)_w}\frac{\partial}{\partial w},$$

so W is globally defined on X. Now we will show that $\mathcal{W}$ satisfies conditions in the theorem.

Since

$$\left\|\varphi^{-1/2}\mathcal{W}\right\|^2_{\omega_X} = \left\|\frac{-2i}{(\log\varphi)_z}\frac{\partial}{\partial z}\right\|^2_{\omega_X} = \frac{4\lambda}{(\log\varphi)_z\,(\log\varphi)_{\bar{z}}} = 1,$$

the vector field $\varphi^{-1/2}\mathcal{W}$ has a unit length with respect to the complete metric ω_X, so the corresponding real vector field $\operatorname{Re}\varphi^{-1/2}\mathcal{W} = \varphi^{-1/2}(\mathcal{W} + \overline{\mathcal{W}})$ is complete. Moreover

$$(\operatorname{Re}\mathcal{W})\varphi = \frac{-2i\varphi^{3/2}}{\varphi_z}\varphi_z + \frac{2i\varphi^{3/2}}{\varphi_{\bar{z}}}\varphi_{\bar{z}} = 0.$$

Hence it remains to show the completeness of $\mathcal{W}$. Take any integral curve $\gamma : \mathbb{R} \to X$ of $\varphi^{-1/2}\operatorname{Re}\mathcal{W}$. It satisfies

$$\left(\varphi^{-1/2}(\operatorname{Re}\mathcal{W})\right)\circ\gamma = \dot{\gamma}$$

equivalently

$$(\operatorname{Re}\mathcal{W})\circ\gamma = \left(\varphi^{1/2}\circ\gamma\right)\dot{\gamma}.$$

The condition $(\operatorname{Re}\mathcal{W})\varphi \equiv 0$, equivalently $\varphi^{-1/2}(\operatorname{Re}\mathcal{W})\varphi \equiv 0$, implies that the curve γ is on a level set of φ so $\varphi^{1/2} \circ \gamma \equiv C$ for some constant C. The curve $\sigma : \mathbb{R} \to X$ defined by $\sigma(t) = \gamma(Ct)$ satisfies

$$(\operatorname{Re}\mathcal{W}) \circ \sigma(t) = (\operatorname{Re}\mathcal{W})(\gamma(Ct)) = C\dot{\gamma}(Ct) = \dot{\sigma}(t).$$

This means that $\sigma : \mathbb{R} \to X$ is the integral curve of $\operatorname{Re}\mathcal{W}$; therefore $\operatorname{Re}\mathcal{W}$ is complete. This completes the proof. $\square$

4. Complete holomorphic vector fields on the unit disc

In this section, we introduce parabolic and hyperbolic vector fields on the unit disc and discuss their relation to the model potential,

$$\varphi_0 = \frac{|1+z|^4}{\left(1-|z|^2\right)^2}, \tag{4.1}$$

where it is φ_θ in (2.1) with $\theta = 0$.

4.1. Nowhere vanishing complete holomorphic vector fields from the left-half plane

On the left-half plane $\mathbf{H} = \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$, there are two kinds of affine transformations:

$$\mathcal{D}_s(w) = e^{2s}w \quad \text{and} \quad \mathcal{T}_s(w) = w + 2is$$

for $s \in \mathbb{R}$. Their infinitesimal generators are

$$\mathcal{D} = 2w\frac{\partial}{\partial w} \quad \text{and} \quad \mathcal{T} = 2i\frac{\partial}{\partial w}$$

which are nowhere vanishing complete holomorphic vector fields of $\mathbf{H}$. Note that

$$(\mathcal{T}_s)_*\mathcal{D} = 2(w-2is)\frac{\partial}{\partial w} = \mathcal{D} - 2s\mathcal{T} \quad \text{and} \quad (\mathcal{T}_s)_*\mathcal{T} = 2i\frac{\partial}{\partial w} = \mathcal{T} \tag{4.2}$$

for any s.

For the Cayley transform $F : \mathbf{H} \to \Delta$ defined by

$$\begin{aligned} F : \mathbf{H} &\longrightarrow \Delta \\ w &\longmapsto z = \frac{1+w}{1-w}, \end{aligned} \tag{4.3}$$

we can take two nowhere vanishing complete holomorphic vector fields of Δ:

$$\mathcal{H} = F_*(\mathcal{D}) = (z^2-1)\frac{\partial}{\partial z}$$

and

$$\mathcal{P} = F_*(\mathcal{T}) = i(z+1)^2\frac{\partial}{\partial z}.$$

When we define $\mathcal{H}_s = F \circ \mathcal{D}_s \circ F^{-1}$ and $\mathcal{P}_s = F \circ \mathcal{T}_s \circ F^{-1}$, vector fields $\mathcal{H}$ and $\mathcal{P}$ are infinitesimal generators of $\mathcal{H}_s$ and $\mathcal{P}_s$, respectively. Moreover Equation (4.2) can be written by

$$(\mathcal{P}_s)_*\mathcal{H} = \mathcal{H} - 2s\mathcal{P} \quad \text{and} \quad (\mathcal{P}_s)_*\mathcal{P} = \mathcal{P}. \tag{4.4}$$

There is another complete holomorphic vector field $\mathcal{R} = iz\partial/\partial z$ generating the rotational symmetry

$$\mathcal{R}_s(z) = e^{is}z. \tag{4.5}$$

The holomorphic automorphism group of Δ is a real 3-dimension connected Lie group (cf. see [1, 9]), we can conclude that any complete holomorphic vector field can be a real linear combination of $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{R}$. Since $\mathcal{H}(-1) = \mathcal{P}(-1) = 0$ and $\mathcal{R}(-1) = -i\partial/\partial z$, we have:

Lemma 4.1. *If $\mathcal{W}$ is a complete holomorphic vector field of Δ satisfying $\mathcal{W}(-1) = 0$, then there exist $a, b \in \mathbb{R}$ with $\mathcal{W} = a\mathcal{H} + b\mathcal{P}$.*

4.2. Hyperbolic vector fields

In this subsection, we will show that the hyperbolic vector field $\mathcal{H}$ can not be tangent to a Kähler potential with a constant length differential.

By the simple computation,

$$\mathcal{H}(\log\varphi_0) = (z^2-1)\frac{2(1+\bar{z})}{(1+z)(1-|z|^2)} = 2\frac{|z|^2 + z - \bar{z} - 1}{(1-|z|^2)},$$

we get

$$(\operatorname{Re}\mathcal{H})\log\varphi_0 \equiv -4.$$

That means $\operatorname{Re}\mathcal{H}$ is nowhere tangent to φ_0. Moreover, we have:

Lemma 4.2. *Let $\varphi : \Delta \to \mathbb{R}$ with $dd^c\log\varphi = 2\omega_\Delta$ and $\|d\log\varphi\|^2_{\omega_\Delta} \equiv 4$. If $(\operatorname{Re}\mathcal{H})\log\varphi \equiv c$ for some c, then $c = \pm 4$.*

Proof. Since $dd^c\log\varphi_0 = 2\omega_\Delta$ also, the function $\log\varphi - \log\varphi_0$ is harmonic; hence we may let $\log\varphi = \log\varphi_0 + f + \bar{f}$ for some holomorphic function $f : \Delta \to \mathbb{C}$. Then the condition $(\operatorname{Re}\mathcal{H})\log\varphi \equiv c$ can be written by

$$(\operatorname{Re}\mathcal{H})\log\varphi = -4 + (z^2-1)f' + (\bar{z}^2-1)\bar{f}' \equiv c. \tag{4.6}$$

This implies that $(z^2-1)f'$ is constant. Thus we can let

$$f' = \frac{C}{z^2-1} \tag{4.7}$$

for some $C \in \mathbb{C}$. Since

$$\frac{\partial}{\partial z}\log\varphi = f' + \frac{\partial}{\partial z}\log\varphi_0 = f' + \frac{2(1+\bar{z})}{(1+z)(1-|z|^2)},$$

we have

$$\|d\log\varphi\|^2_{\omega_\Delta} = \left(\frac{\partial}{\partial z}\log\varphi\right)\left(\frac{\partial}{\partial\bar{z}}\log\varphi\right)\frac{1}{\lambda_\Delta}$$

$$= |f'|^2 (1-|z|^2)^2 + \frac{2(1+\bar{z})(1-|z|^2)}{(1+z)}\bar{f}' + \frac{2(1+z)(1-|z|^2)}{(1+\bar{z})}f' + \|d\log\varphi_0\|^2_{\omega_\Delta}.$$

From the condition $\|d\log\varphi\|^2_{\omega_\Delta} \equiv 4 \equiv \|d\log\varphi_0\|^2_{\omega_\Delta}$, it follows

$$|f'|^2 (1-|z|^2)^2 = -\frac{2(1+\bar{z})(1-|z|^2)}{(1+z)}\bar{f}' - \frac{2(1+z)(1-|z|^2)}{(1+\bar{z})}f',$$

equivalently

$$\frac{1}{2}|f'|^2 (1-|z|^2) = -\frac{(1+\bar{z})}{(1+z)}\bar{f}' - \frac{(1+z)}{(1+\bar{z})}f'. \tag{4.8}$$

Applying (4.7) to the right side above,

$$\begin{aligned} -\frac{(1+\bar{z})}{(1+z)}\bar{f}' - \frac{(1+z)}{(1+\bar{z})}f' &= \frac{(1+\bar{z})}{(1+z)}\frac{\bar{C}}{1-\bar{z}^2} + \frac{(1+z)}{(1+\bar{z})}\frac{C}{1-z^2} \\ &= \frac{(1+\bar{z}-z-|z|^2)\bar{C} + (1-\bar{z}+z-|z|^2)C}{|1-z^2|^2}. \end{aligned}$$

Let $C = a+bi$ for $a, b \in \mathbb{R}$, then

$$(1+\bar{z}-z-|z|^2)\bar{C} + (1-\bar{z}+z-|z|^2)C = 2a(1-|z|^2) + 2bi(z-\bar{z}).$$

Now Equation (4.8) can be written by

$$\frac{1}{2}\frac{|C|^2}{|z^2-1|^2}(1-|z|^2) = \frac{2a(1-|z|^2)+2bi(z-\bar{z})}{|1-z^2|^2},$$

so we have

$$(|C|^2 - 4a)(1-|z|^2) = 4bi(z-\bar{z})$$

on Δ. Take $\partial\bar\partial$ to above, we have

$$|C|^2 - 4a = 0.$$

Simultaneously $b = 0$ so $C = a$. Now we have $a^2 = 4a$. Such a is 0 or 4. If $f' = 4/(z^2-1)$, then $c = 4$ from (4.6). If $f' = 0$, then $c = -4$. □

4.3. Parabolic vector fields

Since

$$\mathcal{P}(\log\varphi_0) = i(z+1)^2 \frac{2(1+\bar{z})}{(1+z)(1-|z|^2)} = 2i\frac{|1+z|^2}{1-|z|^2},$$

we have

$$(\operatorname{Re}\mathcal{P})\log\varphi_0 \equiv 0.$$

That means that the parabolic vector field $\mathcal{P}$ is tangent to φ_0. The vector field $\mathcal{P}$ is indeed the nowhere vanishing complete holomorphic vector field as constructed in Theorem 3.1 corresponding to φ_0. The main result of this section is the following.

Lemma 4.3. *Let $\varphi : \Delta \to \mathbb{R}$ with $dd^c \log\varphi = 2\omega_\Delta$ and $\|d\log\varphi\|^2_{\omega_\Delta} \equiv 4$. If $(\operatorname{Re}\mathcal{P})\log\varphi \equiv c$ for some c, then $c = 0$ and $\varphi = r\varphi_0$ for some $r > 0$.*

Proof. By the same way in the proof of Lemma 4.2, we let $\log\varphi = \log\varphi_0 + f + \bar{f}$ for some holomorphic $f : \Delta \to \mathbb{C}$. Since

$$(\mathrm{Re}\,\mathcal{P})\log\varphi = i(z+1)^2 f' - i(\bar{z}+1)^2 \bar{f}' \equiv c \tag{4.9}$$

it follows that $(z+1)^2 f'$ is constant. Thus we have

$$f' = \frac{C}{(z+1)^2} \tag{4.10}$$

for some $C \in \mathbb{C}$. Since (4.8) also holds, we can apply (4.10) to the right side of (4.8) to get

$$\begin{aligned} -\frac{(1+\bar{z})}{(1+z)}\bar{f}' - \frac{(1+z)}{(1+\bar{z})}f' &= -\frac{(1+\bar{z})}{(1+z)}\frac{\bar{C}}{(\bar{z}+1)^2} - \frac{(1+z)}{(1+\bar{z})}\frac{C}{(z+1)^2} \\ &= \frac{-\bar{C}}{|1+z|^2} + \frac{-C}{|1+z|^2} = \frac{-\bar{C}-C}{|1+z|^2}. \end{aligned}$$

Now Equation (4.8) is can be written by

$$\frac{|C|^2}{|z+1|^4}(1-|z|^2) = 2\frac{-\bar{C}-C}{|1+z|^2}$$

equivalently

$$|C|^2(1-|z|^2) = -\left(2\bar{C}+2C\right)|1+z|^2.$$

Evaluating $z=0$, we have $|C|^2 = -2\bar{C} - 2C$. And taking $\partial\bar{\partial}$ to above, we have $-|C|^2 = -2\bar{C} - 2C$. It follows that $C = 0$ so f is constant. Moreover Equation (4.9) implies that $c = 0$. ☐

5. Proof of the main theorem

Now we prove Theorem 2.2 and Corollary 2.3.

Proof of Theorem 2.2. Let $\varphi : \Delta \to \mathbb{R}$ be a function with

$$dd^c \log\varphi = 2\omega_\Delta \quad \text{and} \quad \|d\log\varphi\|^2_{\omega_\Delta} \equiv 4.$$

By Theorem 3.1, we can take a nowhere vanishing complete holomorphic vector field $\mathcal{W}$ with $(\mathrm{Re}\,\mathcal{W})\varphi \equiv 0$. Since every automorphism of Δ has at least one fixed point on $\overline{\Delta}$ and $\mathcal{W}$ is nowhere vanishing on Δ, any nontrivial automorphism generated by $\mathrm{Re}\,\mathcal{W}$ has no fixed point in Δ and should have a common fixed point p at the boundary $\partial\Delta$. This means p is a vanishing point of $\mathcal{W}$. Consider a rotational symmetry $\mathcal{R}_\theta$ in (4.5) satisfying $\mathcal{R}_\theta(-1) = p$. We will show that $\varphi \circ \mathcal{R}_\theta = r\varphi_0$ where φ_0 is as in (4.1) and $r > 0$. This implies that $\varphi = r\varphi_{-\theta}$.

Now we can simply denote by $\varphi = \varphi \circ \mathcal{R}_\theta$ and $\mathcal{W} = (\mathcal{R}_\theta^{-1})_*\mathcal{W}$. Since -1 is a vanishing point of $\mathcal{W}$, Lemma 4.1 implies

$$\mathcal{W} = a\mathcal{H} + b\mathcal{P}$$

for some real numbers a, b.

Suppose that $a \neq 0$. Equation (4.4) implies that

$$(\mathcal{P}_s)_* \mathcal{W} = (\mathcal{P}_s)_*(a\mathcal{H} + b\mathcal{P}) = a\mathcal{H} - 2as\mathcal{P} + b\mathcal{P} = a\mathcal{H} + (b - 2as)\mathcal{P}.$$

Take $s = b/2a$, then $\widetilde{\mathcal{W}} = (\mathcal{P}_s)_*\mathcal{W} = a\mathcal{H}$. Let $\tilde{\varphi} = \varphi \circ \mathcal{P}_{-s}$ for this s. Then $\tilde{\varphi}$ satisfies conditions in Theorem 2.2 and $(\mathrm{Re}\,\widetilde{\mathcal{W}})\tilde{\varphi} \equiv 0$. But Lemma 4.2 said that $(\mathrm{Re}\,\widetilde{\mathcal{W}})\tilde{\varphi} = a(\mathrm{Re}\,\mathcal{H})\tilde{\varphi} \equiv \pm 4a\tilde{\varphi}$. It contradicts to $(\mathrm{Re}\,\mathcal{W})\varphi \equiv 0$ equivalently $(\mathrm{Re}\,\widetilde{\mathcal{W}})\tilde{\varphi} \equiv 0$. Thus $a = 0$.

Now $\mathcal{W} = b\mathcal{P}$. Since $\mathcal{W}$ is nowhere vanishing already, $b \neq 0$. The condition $(\mathrm{Re}\,\mathcal{W})\varphi \equiv 0$ implies $(\mathrm{Re}\,\mathcal{P})\varphi \equiv 0$. Lemma 4.3 says that $\varphi = r\varphi_0$ for some positive r. This completes the proof. □

Proof of Corollary 2.3. Let D be a simply connected proper domain in $\mathbb{C}$ and let $\omega_D = i\lambda_D dz \wedge d\bar{z}$ be its Poincaré metric with $\|d\log\lambda_D\|^2_{\omega_D} \equiv 4$. By Theorem 3.1, there is a nowhere vanishing complete holomorphic vector field $\mathcal{W}$ with $(\mathrm{Re}\,\mathcal{W})\lambda_D \equiv 0$. Take a biholomorphism $G : \Delta \to D$ and let

$$\varphi = \lambda_D \circ G \quad \text{and} \quad \mathcal{Z} = (G^{-1})_*\mathcal{W}\,.$$

Note that $(\mathrm{Re}\,\mathcal{Z})\varphi \equiv 0$ by assumption. Using the rotational symmetry $\mathcal{R}_\theta$ of Δ which is also affine, we may assume that $\mathcal{Z}(-1) = 0$ and we will prove that G is a Cayley transform.

Since $G : (\Delta, \omega_\Delta) \to (D, \omega_D)$ is an isometry, we have $G^*\omega_D = \omega_\Delta$, equivalently

$$\varphi = \frac{\lambda_\Delta}{|G'|^2}.$$

Moreover $d\log\varphi = d(G^*\log\lambda_D)$ implies that $\|d\log\varphi\|^2_{\omega_D} = \|d(G^*\log\lambda_D)\|^2_{\omega_D} \equiv 4$. By Theorem 2.2, we have

$$\frac{\lambda_\Delta}{|G'|^2} = \varphi = r\varphi_0 = r\lambda_\Delta\,|1 + z|^4$$

for some positive r. This means that $G' = e^{i\theta'}/\sqrt{r}(1 + z)^2$ for some $\theta' \in \mathbb{R}$ so that

$$G = \frac{e^{i\theta'}}{2\sqrt{r}}\frac{z - 1}{z + 1} + C.$$

Since the function $z \mapsto (z - 1)/(z + 1)$ is the inverse mapping of the Cayley transform $F : \mathbf{H} \to \Delta$ in (4.3), we have

$$\begin{aligned} G \circ F : \mathbf{H} &\to D \\ z &\mapsto \frac{e^{i\theta'}}{2\sqrt{r}}z + C. \end{aligned}$$

This implies that $D = G(F(\mathbf{H}))$ is affine equivalent to $\mathbf{H}$. □

References

[1] H. Cartan, *Sur les groupes de transformations analytiques*, Actualités scientifiques et industrielles: Exposés mathématiques. Hermann et cie, 1935.

[2] H. Donnelly, *L_2 cohomology of pseudoconvex domains with complete Kähler metric*, Michigan Math. J. **41** (1994), no. 3, 433–442.

[3] ———, *L_2 cohomology of the Bergman metric for weakly pseudoconvex domains*, Illinois J. Math. **41** (1997), no. 1, 151–160. `http://projecteuclid.org/euclid.ijm/1255985851`

[4] H. Donnelly and C. Fefferman, *L^2-cohomology and index theorem for the Bergman metric*, Ann. of Math. (2) **118** (1983), no. 3, 593–618. `https://doi.org/10.2307/2006983`

[5] S. G. Gindikin, *Analysis in homogeneous domains*, Uspehi Mat. Nauk **19** (1964), no. 4 (118), 3–92.

[6] M. Gromov, *Kähler hyperbolicity and L_2-Hodge theory*, J. Differential Geom. **33** (1991), no. 1, 263–292.

[7] C. Kai and T. Ohsawa, *A note on the Bergman metric of bounded homogeneous domains*, Nagoya Math. J. **186** (2007), 157–163. `https://doi.org/10.1017/S0027763000009399`

[8] W. Kaup, Y. Matsushima, and T. Ochiai, *On the automorphisms and equivalences of generalized Siegel domains*, Amer. J. Math. **92** (1970), 475–498. `https://doi.org/10.2307/2373335`

[9] R. Narasimhan, *Several Complex Variables*, The University of Chicago Press, Chicago, IL, 1971.

[10] È. B. Vinberg, S. G. Gindikin, and I. I. Pjateckiĭ-Šapiro, *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moskov. Mat. Obšč. **12** (1963), 359–388.

Young-Jun Choi
Department of Mathematics
Pusan National University
Busan 46241, Korea
Email address: `youngjun.choi@pusan.ac.kr`

Kang-Hyurk Lee
Department of Mathematics and Research Institute of Natural Science
Gyeongsang National University
Jinju 52828, Korea
Email address: `nyawoo@gnu.ac.kr`

Sungmin Yoo
Center for Geometry and Physics
Institute for Basic Science (IBS)
Pohang 37673, Korea
Email address: `sungmin@ibs.re.kr`