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ON THE EXTENSION DIMENSION OF MODULE CATEGORIES

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Abstract. Let Λ be an Artin algebra and $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules. We prove that the radical layer length of Λ is an upper bound for the radical layer length of $\operatorname{mod}\Lambda$. We give an upper bound for the extension dimension of $\operatorname{mod}\Lambda$ in terms of the injective dimension of a certain class of simple right Λ-modules and the radical layer length of $D\Lambda$.

1. Introduction

The dimension of triangulated categories was studied in [2, 9], which measures how quickly the category can be built from one object. This dimension can be used to compute the representation dimension of Artin algebras [7,8,11]. Rouquier proved in [9] that the dimension of the bounded derived category of $\operatorname{mod}\Lambda$ is at most $\min\{\operatorname{gl.dim}\Lambda, \operatorname{LL}(\Lambda)-1\}$, where $\operatorname{gl.dim}\Lambda$ and $\operatorname{LL}(\Lambda)$ are the global dimension and the Loewy length of an Artin algebra Λ, respectively.

Let $\mathcal{A}$ be an abelian category having enough projective objects and enough injective objects. As an analogue of the dimension of triangulated categories, the (extension) dimension $\dim\mathcal{A}$ of an abelian category $\mathcal{A}$ was introduced by Beligiannis in [1], also see [3]. Let Λ be an Artin algebra. The extension dimension $\dim\operatorname{mod}\Lambda$ is also an invariant that measures how far Λ is from having finite representation type. Thus it is an important and meaningful work to look for a suitable upper bound for the extension dimension. It was proved in [1] that $\dim\operatorname{mod}\Lambda \leq \operatorname{LL}(\Lambda)-1$. Zheng, Ma and Huang also gave an upper bound for the extension dimension of $\operatorname{mod}\Lambda$ in terms of the projective dimension of a certain class of simple right Λ-modules and the radical layer length of Λ in [12]. Based on these works, in this paper we will study further properties of the extension dimension of $\operatorname{mod}\Lambda$ in terms of the injective dimension of a certain class of simple right Λ-modules and the radical layer length of Λ, and will give

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a smaller upper bound for the extension dimension of $\operatorname{mod}\Lambda$ which is better than $\mathrm{LL}(\Lambda)-1$ and $\mathrm{gl.dim}\,\Lambda$. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let Λ be an Artin algebra and $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules. In Section 3, we investigate the radical layer length of modules, and get that the radical layer lengths of Λ and $D\Lambda$ are equal. We also prove that the radical layer length of Λ is an upper bound for the radical layer length of $\operatorname{mod}\Lambda$. In Section 4, we give an upper bound for the extension dimension of $\operatorname{mod}\Lambda$ in terms of the injective dimension of a certain class of simple right Λ-modules and the radical layer length of $D\Lambda$, that is,

Theorem 1.1 (Corollary 4.4)**.** *Let $\mathcal{S}$ be a subset of the set of the simple modules with finite projective dimension and finite injective dimension in* $\operatorname{mod}\Lambda$. *Then* $\dim\operatorname{mod}\Lambda \leq \min\{\operatorname{id}\mathcal{S}, \operatorname{pd}\mathcal{S}\} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$.

In Section 5, we give examples to show that there is no necessary relationship between $\operatorname{id}\mathcal{S}+\ell\ell^{t_{\mathcal{S}}}(D\Lambda)$ and $\operatorname{pd}\mathcal{S}+\ell\ell^{t_{\mathcal{S}}}(\Lambda)$.

2. Preliminaries

Let $\mathcal{A}$ be an abelian category. All subcategories of $\mathcal{A}$ are full, additive and closed under isomorphisms and all functors between additive categories are additive. For a subclass $\mathcal{U}$ of $\mathcal{A}$, we use $\operatorname{add}\mathcal{U}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{U}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ be subcategories of $\mathcal{A}$. Define

$$\mathcal{U}_1 \diamond \mathcal{U}_2 := \operatorname{add}\{A \in \mathcal{A} \mid \text{there is an exact sequence } 0 \to U_1 \to A \to U_2 \to 0 \text{ in } \mathcal{A} \text{ with } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}.$$

The category $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$ can be inductively described as follows

$$\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n := \operatorname{add}\{A \in \mathcal{A} \mid \text{there is an exact sequence } 0 \to U \to A \to V \to 0 \text{ in } \mathcal{A} \text{ with } U \in \mathcal{U}_1 \text{ and } V \in \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n\}.$$

For a subcategory $\mathcal{U}$ of $\mathcal{A}$, set $\langle \mathcal{U} \rangle_0 = 0$, $\langle \mathcal{U} \rangle_1 = \operatorname{add}\mathcal{U}$, $\langle \mathcal{U} \rangle_n = \langle \mathcal{U} \rangle_1 \diamond \langle \mathcal{U} \rangle_{n-1}$ for any $n \geq 2$, and $\langle \mathcal{U} \rangle_\infty = \bigcup_{n \geq 0} \langle \mathcal{U} \rangle_n$ ([1]). If T is an object in $\mathcal{A}$, we write $\langle T \rangle_n$ instead of $\langle \{T\} \rangle_n$. For any subcategories $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ of $\mathcal{A}$, by [3, Proposition 2.2] we have

$$(\mathcal{U} \diamond \mathcal{V}) \diamond \mathcal{W} = \mathcal{U} \diamond (\mathcal{V} \diamond \mathcal{W}).$$

Definition 2.1 ([3, Definition 5.2])**.** For any subcategory $\mathcal{X}$ of $\mathcal{A}$, one defines

$$\mathbf{size}_{\mathcal{A}}\, \mathcal{X} := \inf\{n \geq 0 \mid \mathcal{X} \subseteq \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\},$$

$$\mathbf{rank}_{\mathcal{A}}\, \mathcal{X} := \inf\{n \geq 0 \mid \mathcal{X} = \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\}.$$

The *extension dimension* $\dim\mathcal{A}$ of $\mathcal{A}$ is defined to be $\dim\mathcal{A} := \mathbf{rank}_{\mathcal{A}}\,\mathcal{A}$.

It is easy to see that $\dim\mathcal{A} = \mathbf{rank}_{\mathcal{A}}\,\mathcal{A} = \mathbf{size}_{\mathcal{A}}\,\mathcal{A}$. Also one has the following easy and useful observations.

Proposition 2.2 ([12, Proposition 2.2]). *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be subcategories of $\mathcal{A}$ with $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Then*

(1) *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are subcategories of $\mathcal{A}$ with $\mathcal{V}_1 \subseteq \mathcal{V}_2$, then $\mathcal{U}_1 \diamond \mathcal{V}_1 \subseteq \mathcal{U}_2 \diamond \mathcal{V}_2$;*
(2) *$\langle \mathcal{U}_1 \rangle_n \subseteq \langle \mathcal{U}_2 \rangle_n$ for any $n \geq 1$;*
(3) *$\langle \mathcal{U}_1 \rangle_n \subseteq \langle \mathcal{U}_1 \rangle_{n+1}$ for any $n \geq 1$;*
(4) *$\mathbf{size}_{\mathcal{A}} \mathcal{U}_1 \leq \mathbf{size}_{\mathcal{A}} \mathcal{U}_2$.*

For two subcategories $\mathcal{U}, \mathcal{V}$ of $\mathcal{A}$, we set $\mathcal{U} \oplus \mathcal{V} = \{U \oplus V \mid U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Corollary 2.3 ([12, Corollary 2.3]). *For any $T_1, T_2 \in \mathcal{A}$ and $m, n \geq 1$,*

(1) *$\langle T_1 \rangle_m \diamond \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_{m+n}$;*
(2) *$\langle T_1 \rangle_m \oplus \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_{\max\{m,n\}}$.*

3. Layer lengths

We recall some notions from [4]. Let $\mathcal{C}$ be a *length-category*, that is, $\mathcal{C}$ is an abelian, skeletally small category and every object of $\mathcal{C}$ has a finite composition series. We denote by $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ the category of all additive functors from $\mathcal{C}$ to $\mathcal{C}$, and denote by rad the Jacobson radical lying in $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$. Let $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ and α be a subfunctor of β, we have the quotient functor $\beta/\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ which is defined as follows.

(1) $(\beta/\alpha)(M) := \beta(M)/\alpha(M)$ for any $M \in \mathcal{C}$;
(2) $(\beta/\alpha)(f)$ is the induced quotient morphism: for any $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \alpha(M) & \longrightarrow & \beta(M) & \longrightarrow & \beta(M)/\alpha(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha(f)} & & \downarrow{\scriptstyle \beta(f)} & & \downarrow{\scriptstyle (\beta/\alpha)(f)} & & \\ 0 & \longrightarrow & \alpha(N) & \longrightarrow & \beta(N) & \longrightarrow & \beta(N)/\alpha(N) & \longrightarrow & 0. \end{array}$$

For a subfunctor $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ of the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$, we write $q_\alpha := 1_{\mathcal{C}}/\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$. For any $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, set the *$\alpha$-radical functor* $F_\alpha := \mathrm{rad} \circ \alpha$ and the *α-socle quotient functor* $G_\alpha := \alpha/(\mathrm{soc} \circ \alpha)$.

Definition 3.1 ([4, Definition 3.1]). For any $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, we define the *$(\alpha, \beta)$-layer length* $\ell\ell_\alpha^\beta : \mathcal{C} \longrightarrow \mathbb{N} \cup \{\infty\}$ via $\ell\ell_\alpha^\beta(M) = \min\{i \geq 0 \mid \alpha \circ \beta^i(M) = 0\}$; and the *$\alpha$-radical layer length* $\ell\ell^\alpha := \ell\ell_\alpha^{F_\alpha}$ and the *α-socle layer length* $\ell\ell_\alpha := \ell\ell_\alpha^{G_\alpha}$.

Note that, if $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ is either a subfunctor or a quotient functor of $1_{\mathcal{C}}$, then $\ell\ell_\alpha(M)$ and $\ell\ell^\alpha(M)$ are finite for all M in $\mathcal{C}$. And the Loewy length is obtained by taking $\alpha = 1_{\mathcal{C}}$ in Definition 3.1.

Recall that a *torsion pair* (or *torsion theory*) for $\mathcal{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ of objects in $\mathcal{C}$ satisfying the following conditions.

(1) $\mathrm{Hom}_{\mathcal{C}}(M, N) = 0$ for any $M \in \mathcal{T}$ and $N \in \mathcal{F}$;
(2) an object $X \in \mathcal{C}$ is in $\mathcal{T}$ if $\mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{F}} = 0$;
(3) an object $Y \in \mathcal{C}$ is in $\mathcal{F}$ if $\mathrm{Hom}_{\mathcal{C}}(-, Y)|_{\mathcal{T}} = 0$.

Let $(\mathcal{T},\mathcal{F})$ be a torsion pair for $\mathcal{C}$. Recall that $t := \mathrm{Trace}_{\mathcal{T}}$ is the so called *torsion radical* attached to $(\mathcal{T},\mathcal{F})$. Then $t(M) := \Sigma\{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_{\mathcal{C}}(T, M)$ with $T \in \mathcal{T}\}$ is the largest subobject of M lying in $\mathcal{T}$.

Definition 3.2 ([6])**.** A class $\mathcal{X}$ in $\mathcal{C}$ is called a *ttf-class* if there exist classes $\mathcal{T}$ and $\mathcal{F}$ such that $(\mathcal{T},\mathcal{X})$ and $(\mathcal{X},\mathcal{F})$ are torsion theories for $\mathcal{C}$. In this case, the triple $(\mathcal{T},\mathcal{X},\mathcal{F})$ is called a *ttf-theory*.

In the following sections, Λ is an Artin algebra. Then the category $\mathrm{mod}\,\Lambda$ of finite generated right Λ-modules is a length-category. We use $\mathcal{S}^{<\infty}$ to denote the set of the simple modules in $\mathrm{mod}\,\Lambda$ with finite injective dimension. From now on, assume that $\mathcal{S}$ is a subset of $\mathcal{S}^{<\infty}$ and $\mathcal{S}'$ is the set of all others simple modules in $\mathrm{mod}\,\Lambda$. We write $\mathfrak{F}(\mathcal{S}) := \{M \in \mathrm{mod}\,\Lambda \mid$ there exists a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that each quotient M_i/M_{i-1} is isomorphic to some module in $\mathcal{S}\}$.

Lemma 3.3 ([4, Lemma 5.7 and Proposition 5.9])**.** *Let $\mathcal{S}$ be some set of simple objects in $\mathrm{mod}\,\Lambda$, $\mathcal{S}'$ be all others simple objects in $\mathrm{mod}\,\Lambda$ and $\mathfrak{F}(\mathcal{S})$ be as above. Then $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}), \mathcal{F}_{\mathcal{S}})$ is a ttf-theory where $\mathcal{T}_{\mathcal{S}} = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{top}\, M \in \mathrm{add}\,\mathcal{S}'\}$ and $\mathcal{F}_{\mathcal{S}} = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{soc}\, M \in \mathrm{add}\,\mathcal{S}'\}$.*

By [4, Proposition 5.3], we denote the torsion radical $t_{\mathcal{S}} = \mathrm{Trace}_{\mathcal{T}_{\mathcal{S}}}$ and $\tilde{t_{\mathcal{S}}} = \mathrm{Trace}_{\mathfrak{F}(\mathcal{S})}$. Here $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}), \mathcal{F}_{\mathcal{S}})$ is a ttf-theory in $\mathrm{mod}\,\Lambda$, and we have the following

Proposition 3.4.

(1) *The functor $t_{\mathcal{S}}$ preserves monomorphisms and epimorphisms*;
(2) *The functor $q_{\tilde{t_{\mathcal{S}}}}$ preserves monomorphisms and epimorphisms*;
(3) *The functor $G_{q_{\tilde{t_{\mathcal{S}}}}}$ preserves monomorphisms and epimorphisms.*

Proof. (1) Let $f : M \longrightarrow N$ be a monomorphism. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & t_{\mathcal{S}}(M) & \longrightarrow & M & \longrightarrow & q_{t_{\mathcal{S}}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t_{\mathcal{S}}(f)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q_{t_{\mathcal{S}}}(f)} & & \\
0 & \longrightarrow & t_{\mathcal{S}}(N) & \longrightarrow & N & \longrightarrow & q_{t_{\mathcal{S}}}(N) & \longrightarrow & 0.
\end{array}$$

By the commutativity of left square, we get that $t_{\mathcal{S}}(f)$ is a monomorphism. Since $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}), \mathcal{F}_{\mathcal{S}})$ is a ttf-theory, then $t_{\mathcal{S}}$ is torsion radical and $\mathfrak{F}(\mathcal{S})$ is closed under quotients. By [10, Ch.VI.Ex 5], we get that $t_{\mathcal{S}}$ preserves epimorphisms.

(2) Let $f : M \longrightarrow N$ be an epimorphism. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{t_{\mathcal{S}}}(M) & \longrightarrow & M & \longrightarrow & q_{\tilde{t_{\mathcal{S}}}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{t_{\mathcal{S}}}(f)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q_{\tilde{t_{\mathcal{S}}}}(f)} & & \\
0 & \longrightarrow & \tilde{t_{\mathcal{S}}}(N) & \longrightarrow & N & \longrightarrow & q_{\tilde{t_{\mathcal{S}}}}(N) & \longrightarrow & 0. \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

By the commutativity of right square, we get that $q_{\tilde{t}_{\mathcal{S}}}(f)$ is an epimorphism. Because $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}), \mathcal{F}_{\mathcal{S}})$ is a ttf-theory, then $\tilde{t_{\mathcal{S}}}$ is torsion radical and $\mathfrak{F}(\mathcal{S})$ is closed under subobjects. Hence $q_{\tilde{t_{\mathcal{S}}}}$ preserves monomorphisms by [4, Lemma 3.7].

(3) The functor $G_{q_{\tilde{t_{\mathcal{S}}}}}$ preserves monomorphisms since $q_{\tilde{t_{\mathcal{S}}}}$ preserves monomorphisms and $G := 1/\operatorname{soc}$ preserves monomorphisms by [4, Section 3].

Let $f : M \longrightarrow N$ be an epimorphism. By (2), $q_{\tilde{t_{\mathcal{S}}}}(f)$ is an epimorphism. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{soc} q_{\tilde{t_{\mathcal{S}}}}(M) & \longrightarrow & q_{\tilde{t_{\mathcal{S}}}}(M) & \longrightarrow & G_{q_{\tilde{t_{\mathcal{S}}}}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \operatorname{soc} q_{\tilde{t_{\mathcal{S}}}}(f)} & & \downarrow{\scriptstyle q_{\tilde{t_{\mathcal{S}}}}(f)} & & \downarrow{\scriptstyle G_{q_{\tilde{t_{\mathcal{S}}}}}(f)} & & \\
0 & \longrightarrow & \operatorname{soc} q_{\tilde{t_{\mathcal{S}}}}(N) & \longrightarrow & q_{\tilde{t_{\mathcal{S}}}}(N) & \longrightarrow & G_{q_{\tilde{t_{\mathcal{S}}}}}(N) & \longrightarrow & 0. \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

By the commutativity of right square, we get that $G_{q_{\tilde{t_{\mathcal{S}}}}}(f)$ is an epimorphism. $\square$

In the following, we show that the radical layer lengths of Λ and $D\Lambda$ are equal, and the radical layer length of Λ is an upper bound for the radical layer length of $\operatorname{mod}\Lambda$.

Proposition 3.5. *Let Λ be an Artin algebra and $\mathcal{S}$ be some subset of simple objects in $\operatorname{mod}\Lambda$. Then we have*

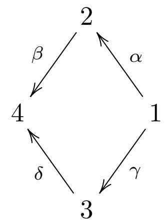
(1) *If $M \in \operatorname{mod}\Lambda$, then $\ell\ell^{t_{\mathcal{S}}}(M) \leq \ell\ell^{t_{\mathcal{S}}}(\Lambda)$;*
(2) $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \ell\ell^{t_{\mathcal{S}}}(D\Lambda) = \ell\ell^{t_{\mathcal{S}}}(\Lambda \oplus D\Lambda)$.

Proof. (1) Since Λ is an Artin algebra and $M \in \operatorname{mod}\Lambda$, we have an epimorphism $\Lambda^n \to M \to 0$. We have $\ell\ell^{t_{\mathcal{S}}}(M) \leq \ell\ell^{t_{\mathcal{S}}}(\Lambda)$ by [4, Lemma 3.4].

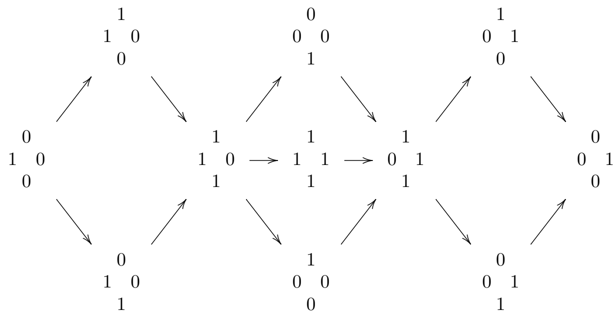
(2) We only prove that $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$. Since Λ is an Artin algebra and $D\Lambda$ is finite generated as Λ-module, we have an epimorphism $\Lambda^n \to D\Lambda \to 0$. Due to $t_{\mathcal{S}}$ and rad preserve epimorphisms, we have $\ell\ell^{t_{\mathcal{S}}}(D\Lambda) \leq \ell\ell^{t_{\mathcal{S}}}(\Lambda)$ by [4, Lemma 3.4]. Similarly, we have a monomorphism $0 \to \Lambda \to (D\Lambda)^n$. Due to $t_{\mathcal{S}}$

and rad preserve monomorphisms, we have $\ell\ell^{t_{\mathcal{S}}}(\Lambda) \leq \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$ by [4, Lemma 3.4]. Hence we have $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$. $\square$

Example 3.6. Consider the algebra Λ given by the quiver



with the relation $\beta\alpha = \delta\gamma$. Then the Auslander-Reiten quiver of $\Gamma(\mathrm{mod}\,\Lambda)$ is of the form



Let $\mathcal{S} := \{S(3), S(4)\}$ and $\mathcal{S}' = \{S(1), S(2)\}$ in $\mathrm{mod}\,\Lambda$. In order to compute $\ell\ell^{t_{\mathcal{S}}}(M)$ for $M \in \mathrm{mod}\,\Lambda$, we need to find the smallest non-negative integer j such that $t_{\mathcal{S}}F^{j}_{t_{\mathcal{S}}}(M) = 0$. Since $\mathrm{top}\,P(1) = S(1) \in \mathrm{add}\,\mathcal{S}'$, we have $t_{\mathcal{S}}(P(1)) = P(1)$ by [4, Proposition 5.9(a)]. Thus

$$F_{t_{\mathcal{S}}}(P(1)) = \mathrm{rad}\,t_{\mathcal{S}}(P(1)) = \mathrm{rad}(P(1)) = \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}$$

$$F_{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}\right) = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \quad t_{\mathcal{S}}F_{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}\right) = 0.$$

Hence $\ell\ell^{t_{\mathcal{S}}}(P(1)) = 2$. Similarly, we have

$$\ell\ell^{t_{\mathcal{S}}}(P(i)) = \begin{cases} 2, & \text{if } i = 1; \\ 1, & \text{if } i = 2; \\ 0, & \text{if } i = 3, 4; \end{cases} \quad \ell\ell^{t_{\mathcal{S}}}(I(i)) = \begin{cases} 1, & \text{if } i = 1, 3; \\ 2, & \text{if } i = 2, 4. \end{cases}$$

Because $\Lambda = \oplus_{i=1}^{4} P(i)$ and $D\Lambda = \oplus_{i=1}^{4} I(i)$, we have

$$\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(P(i)) \mid 1 \leq i \leq 4\} = 2,$$
$$\ell\ell^{t_{\mathcal{S}}}(D\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(I(i)) \mid 1 \leq i \leq 4\} = 2$$

by [4, Lemma 3.4(a)]. Hence $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$. Similarly, we have

$$\ell\ell^{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}\right) = 1, \quad \ell\ell^{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right) = 0, \quad \ell\ell^{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}\right) = 1, \quad \ell\ell^{t_{\mathcal{S}}}\left(\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}\right) = 2.$$

Hence we have $\ell\ell^{t_{\mathcal{S}}}(M) \leq \ell\ell^{t_{\mathcal{S}}}(\Lambda) = \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$ for $M \in \operatorname{mod}\Lambda$.

4. Dimension of module categories

Let $X \in \operatorname{mod}\Lambda$. If there exists a monomorphism $f : X \longrightarrow E$ in $\operatorname{mod}\Lambda$ such that E is an injective envelope of X, then we write $\Omega^{-1}(X) =: \operatorname{Coker} f$. Dually, if there exists an epimorphism $g : P \longrightarrow X$ in $\operatorname{mod}\Lambda$ such that P is a projective cover of X, then we write $\Omega^{1}(X) =: \operatorname{Ker} g$. Inductively, for any $n \geq 2$, we write $\Omega^{n}(X) := \Omega^{1}(\Omega^{n-1}(X))$ and $\Omega^{-n}(X) := \Omega^{-1}(\Omega^{-(n-1)}(X))$.

Proposition 4.1. *Let $\mathcal{S}$ be some subset of simple objects in $\operatorname{mod}\Lambda$ and $M \in \operatorname{mod}\Lambda$. If $q_{\tilde{t}_{\mathcal{S}}}(M) \neq 0$, then $\ell\ell_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}q_{\tilde{t}_{\mathcal{S}}}(M)) \leq \ell\ell^{t_{\mathcal{S}}}(D\Lambda) - 1$.*

Proof. Assume that $q_{\tilde{t}_{\mathcal{S}}}(M) \neq 0$. Consider the following exact sequence

$$0 \longrightarrow q_{\tilde{t}_{\mathcal{S}}}(M) \longrightarrow I \longrightarrow \Omega^{-1}q_{\tilde{t}_{\mathcal{S}}}(M) \longrightarrow 0$$

where I is the injective envelope of $q_{\tilde{t}_{\mathcal{S}}}(M)$. Hence we have $\operatorname{soc} I \subseteq q_{\tilde{t}_{\mathcal{S}}}(M)$. So we have an exact sequence $0 \to q_{\tilde{t}_{\mathcal{S}}}(M)/\operatorname{soc} I \to I/\operatorname{soc} I \to \Omega^{-1}q_{\tilde{t}_{\mathcal{S}}}(M) \to 0$. Since $q_{\tilde{t}_{\mathcal{S}}}$ and $G_{q_{\tilde{t}_{\mathcal{S}}}}$ preserve epimorphisms, we have

$$\ell\ell_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}q_{\tilde{t}_{\mathcal{S}}}(M)) \leq \ell\ell_{q_{\tilde{t}_{\mathcal{S}}}}(I/\operatorname{soc} I)$$

by [4, Lemma 3.4]. Since $\operatorname{soc} I \subseteq q_{\tilde{t}_{\mathcal{S}}}(M)$, $\operatorname{soc} I \in \operatorname{add}\mathcal{S}'$. Hence $\operatorname{top}\operatorname{soc} I \in \operatorname{add}\mathcal{S}'$ and $\operatorname{soc} I \in \mathcal{T}_{\mathcal{S}}$ by [4, Proposition 5.9]. By [4, Corollary 5.6 and Proposition 4.1], $\ell\ell_{q_{\tilde{t}_{\mathcal{S}}}}(I/\operatorname{soc} I) = \ell\ell^{t_{\mathcal{S}}}(I/\operatorname{soc} I) = \ell\ell^{t_{\mathcal{S}}}(I) - 1 \leq \ell\ell^{t_{\mathcal{S}}}(D\Lambda) - 1$. □

Lemma 4.2. *Let $\mathcal{S}$ be a subset of the set $\mathcal{S}^{<\infty}$ of the simple modules in $\operatorname{mod}\Lambda$ with finite injective dimension. If $q_{\tilde{t}_{\mathcal{S}}}(M) \neq 0$ and $\operatorname{id}\mathcal{S} = \alpha$, then we have an isomorphism*

$$\Omega^{\alpha+2}(\Omega^{-(\alpha+2)}(M)) \cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t}_{\mathcal{S}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M)))));$$

and an exact sequence

$$\begin{aligned} 0 \to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t}_{\mathcal{S}}} G^{i-1}_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M))))) \to \\ \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t}_{\mathcal{S}}} G^{i-1}_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M))))) \oplus \Omega^{\alpha+2}(I_i) \oplus P_i \to \\ \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^{i}_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M))))) \to 0 \end{aligned}$$

for any $1 \leq i \leq n-2$. Moreover, we have isomorphisms

$$\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^{i}_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M))))) \cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t}_{\mathcal{S}}} G^{i}_{q_{\tilde{t}_{\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t}_{\mathcal{S}}}(M)))))$$

for any $1 \le i \le n-2$, *and*

$$\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_S}} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))))) \oplus \Omega^{\alpha+2}(I_{n-1}) \cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_S}} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))))).$$

Proof. We have the following exact sequences:

$$0 \to \tilde{t_S}(M) \to M \to q_{\tilde{t_S}}(M) \to 0,$$

$$0 \to \tilde{t_S}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to \Omega^{-1}(q_{\tilde{t_S}}(M)) \to q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0,$$

$$0 \to \operatorname{soc} q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0,$$

$$0 \to \tilde{t_S} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0,$$

$$0 \to \operatorname{soc} q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to G^2_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0,$$

$$\vdots$$

$$0 \to \tilde{t_S} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to q_{\tilde{t_S}} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0,$$

$$0 \to \operatorname{soc} q_{\tilde{t_S}} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to q_{\tilde{t_S}} G^{n-2}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to G^{n-1}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \to 0.$$

By Proposition 4.1, $\ell\ell_{q_{\tilde{t_S}}}(\Omega^{-1} q_{\tilde{t_S}}(M)) \le \ell\ell^{t_S}(D\Lambda) - 1 = n-1$. Hence

$$q_{\tilde{t_S}}(G^{n-1}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) = 0.$$

Then by [4, Proposition 5.3], we have $\operatorname{id} G^{n-1}_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M))) \le \alpha$. We have the following:

$$\Omega^{-(\alpha+2)}(M) \cong \Omega^{-(\alpha+2)}(q_{\tilde{t_S}}(M)),$$

$$\Omega^{-(\alpha+2)}(q_{\tilde{t_S}}(M)) = \Omega^{-(\alpha+1)}(\Omega^{-1} q_{\tilde{t_S}}(M)) \cong \Omega^{-(\alpha+1)}(q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))),$$

$$0 \to \Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \to \Omega^{-(\alpha+1)}(q_{\tilde{t_S}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \oplus I_1 \to \Omega^{-(\alpha+1)}(G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \to 0, \text{ (exact)}$$

$$\Omega^{-(\alpha+1)}(G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \cong \Omega^{-(\alpha+1)}(q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))),$$

$$0 \to \Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \to \Omega^{-(\alpha+1)}(q_{\tilde{t_S}} G_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \oplus I_2 \to \Omega^{-(\alpha+1)}(G^2_{q_{\tilde{t_S}}}(\Omega^{-1}(q_{\tilde{t_S}}(M)))) \to 0, \text{ (exact)}$$

$$\vdots$$

$$\Omega^{-(\alpha+1)}(G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))) \cong \Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))),$$
$$\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))) \oplus I_{n-1}$$
$$\cong \Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))),$$

where all I_i are injective in $\operatorname{mod}\Lambda$; we also have the following

$$\Omega^{\alpha+2}(\Omega^{-(\alpha+2)}(M)) \cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Omega^{-1} q_{\tilde{t_\mathcal{S}}}(M)))$$
$$= \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))),$$
$$0 \to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_\mathcal{S}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))))$$
$$\to \Omega^{(\alpha+2)}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))) \oplus \Omega^{\alpha+2}(I_1) \oplus P_1$$
$$\to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))) \to 0, \ \text{(exact)}$$
$$\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))))$$
$$\cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))),$$
$$0 \to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_\mathcal{S}}} G_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))))$$
$$\to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))) \oplus \Omega^{\alpha+2}(I_2) \oplus P_2$$
$$\to \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^2_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))) \to 0, \ \text{(exact)}$$
$$\vdots$$
$$\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M)))))$$
$$\cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))),$$
$$\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))) \oplus \Omega^{\alpha+2}(I_{n-1})$$
$$\cong \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{\tilde{t_\mathcal{S}}} G^{n-2}_{q_{\tilde{t_\mathcal{S}}}}(\Omega^{-1}(q_{\tilde{t_\mathcal{S}}}(M))))),$$

where all P_i are projective in $\operatorname{mod}\Lambda$. □

In the following, we will show that the sum of the injective dimension of a certain class of simple right Λ-modules and the radical layer length of $D\Lambda$ provides an upper bound for the extension dimension of $\operatorname{mod}\Lambda$.

Theorem 4.3. *Let $\mathcal{S}$ be a subset of the set $\mathcal{S}^{<\infty}$ of the simple modules in* $\operatorname{mod}\Lambda$ *with finite injective dimension. Then* $\dim \operatorname{mod}\Lambda \leq \operatorname{id}\mathcal{S} + \ell\ell^{t_\mathcal{S}}(D\Lambda)$ *where* $\operatorname{id}\mathcal{S} = \sup\{\operatorname{id} M \mid M \in \mathcal{S}\}$ *with* $\mathcal{S} \neq \emptyset$; $\operatorname{id}\mathcal{S} = -1$ *with* $\mathcal{S} = \emptyset$.

Proof. Let $\ell\ell^{t_\mathcal{S}}(D\Lambda) = n$ and $\operatorname{id}\mathcal{S} = \alpha$.

If $n = 0$, that is, $t_\mathcal{S}(D\Lambda) = 0$, then $D\Lambda \in \mathfrak{F}(\mathcal{S})$, which implies that $\mathcal{S}$ is the set of all simple modules. Thus $\mathcal{S} = \mathcal{S}^{<\infty}$ and $\operatorname{gl.dim}\Lambda = \alpha$. So the assertion follows from [12, Corollary 3.6].

Now let $n \geq 1$ and $M \in \operatorname{mod}\Lambda$. Consider the following exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{\alpha+1} \longrightarrow \Omega^{-(\alpha+2)}(M) \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ with all E_i injective. By [3, Lemma 5.8],

$$M \in \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+2)}(M))\rangle_1 \diamond \langle \oplus_{i=0}^{\alpha+1}\Omega^i(D\Lambda)\rangle_{\alpha+2}.$$

By Lemma 4.2 we have

$$\begin{aligned}
&\Omega^{\alpha+2}(\Omega^{-(\alpha+2)}(M))\\
\cong\ &\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\\
\cong\ &\Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{t_{\tilde{S}}}(\Omega^{-1}(q_{t_{\tilde{S}}}(M)))))\\
\in\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{t_{\tilde{S}}}(\Omega^{-1}(q_{t_{\tilde{S}}}(M)))))\rangle_1\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
=\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{t_{\tilde{S}}}(\Omega^{-1}(q_{t_{\tilde{S}}}(M)))))\rangle_1\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{t_{\tilde{S}}}G_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
\subseteq\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{t_{\tilde{S}}}G_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
\subseteq\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{t_{\tilde{S}}}G_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^2_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
\subseteq\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^2_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
&\qquad\qquad\vdots\\
\subseteq\ &\underbrace{\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \cdots \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1}_{n-2}\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(G^{n-2}_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
=\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{t_{\tilde{S}}}G^{n-2}_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
\subseteq\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(q_{t_{\tilde{S}}}G^{n-2}_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M)))) \oplus \Omega^{\alpha+2}(I_{n-1})\rangle_1\\
=\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
&\quad \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\operatorname{soc} q_{t_{\tilde{S}}}G^{n-2}_{q_{t_{\tilde{S}}}}\Omega^{-1}(q_{t_{\tilde{S}}}(M))))\rangle_1\\
\subseteq\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2} \diamond \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_1\\
=\ &\langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-1},
\end{aligned}$$

and hence

$$\begin{aligned} M &\in \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+2)}(M))\rangle_1 \diamond \langle \oplus_{i=0}^{\alpha+1}\Omega^i(D\Lambda)\rangle_{\alpha+2} \\ &\subseteq \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-1} \diamond \langle \oplus_{i=0}^{\alpha+1}\Omega^i(D\Lambda)\rangle_{\alpha+2} \\ &\subseteq \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda)) \oplus (\oplus_{i=0}^{\alpha+1}\Omega^i(D\Lambda))\rangle_{\alpha+1+n}. \text{ (by Corollary 2.3(1))} \end{aligned}$$

It follows that

$$\operatorname{mod}\Lambda = \langle \Omega^{\alpha+2}(\Omega^{-(\alpha+1)}(\Lambda/\operatorname{rad}\Lambda)) \oplus (\oplus_{i=0}^{\alpha+1}\Omega^i(D\Lambda))\rangle_{\alpha+1+n}$$

and $\dim\Lambda \le \alpha + n$. □

Corollary 4.4. *Let $\mathcal{S}$ be a subset of the set of the simple modules with finite projective dimension and finite injective dimension in* $\operatorname{mod}\Lambda$. *Then*

$$\dim\operatorname{mod}\Lambda \le \min\{\operatorname{id}\mathcal{S}, \operatorname{pd}\mathcal{S}\} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda).$$

Proof. It is a direct consequence of Theorem 4.3, [12, Theorem 3.19], and Proposition 3.5(2). □

Corollary 4.5.

(1) ([1, Example 1.6(ii)]) $\dim\operatorname{mod}\Lambda \le \operatorname{LL}(D\Lambda) - 1 = \operatorname{LL}(\Lambda) - 1$;

(2) ([12, Corollary 3.6] and [5, 4.5.1(3)]) $\dim\operatorname{mod}\Lambda \le \operatorname{gl.dim}\Lambda$.

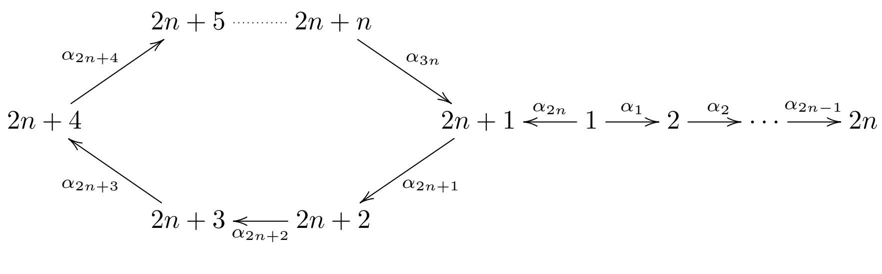
Proof. (1) Let $\mathcal{S} = \emptyset$. Then $\operatorname{id}\mathcal{S} = -1$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (\operatorname{mod}\Lambda, 0)$. By [4, Proposition 5.9(a)], we have $t_{\mathcal{S}}(D\Lambda) = D\Lambda$ and $\ell\ell^{t_{\mathcal{S}}}(D\Lambda) = \operatorname{LL}(D\Lambda)$. It follows from Theorem 4.3 that $\dim\operatorname{mod}\Lambda \le \operatorname{LL}(D\Lambda)-1 = \operatorname{LL}(\Lambda)-1$.

(2) Let $\mathcal{S} = \mathcal{S}^{<\infty} = \{\text{all simple modules in} \operatorname{mod}\Lambda\}$. Then $\operatorname{id}\mathcal{S} = \operatorname{gl.dim}\Lambda$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (0, \operatorname{mod}\Lambda)$. By [4, Proposition 5.3], we have $t_{\mathcal{S}}(D\Lambda) = 0$ and $\ell\ell^{t_{\mathcal{S}}}(D\Lambda) = 0$. It follows from Theorem 4.3 that $\dim\operatorname{mod}\Lambda \le \operatorname{gl.dim}\Lambda$. □

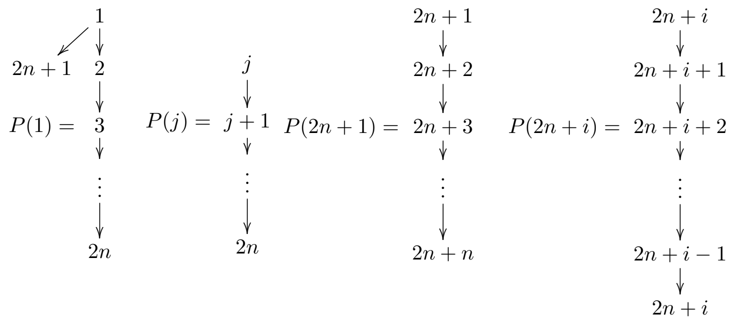
5. Examples

Now we give examples to show that there is no necessary relationship between $\operatorname{id}\mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$ and $\operatorname{pd}\mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda)$. The following first example shows that $\dim\operatorname{mod}\Lambda \le \operatorname{id}\mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda) = \operatorname{pd}\mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda)$.

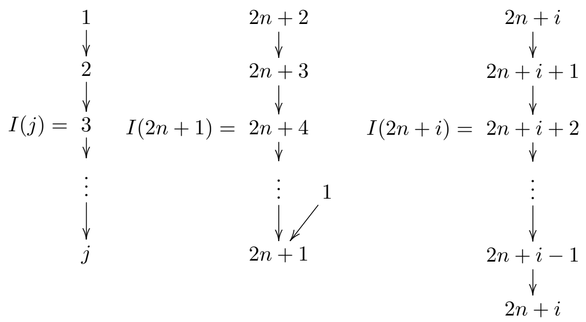
Example 5.1. Consider the bound quiver algebra $\Lambda = kQ/I$, where k is a field and Q is given by



and I is generated by $\{\alpha_{2n}\alpha_{2n+1}; \omega_i$ for $1 \le i \le n-1\}$ where $\omega_1 = \alpha_{2n+1}\alpha_{2n+2} \cdots \alpha_{2n+n}$; $\omega_i = \alpha_{2n+i}\alpha_{2n+i+1}\alpha_{2n+i+2} \cdots \alpha_{2n+i-1}\alpha_{2n+i}$ for $2 \le i \le n-1$. Then the indecomposable projective Λ-modules are



where $2 \le j \le 2n$, $2 \le i \le n$. Then the indecomposable injective Λ-modules are



where $1 \le j \le 2n$, $2 \le i \le n$.

We have

$$\operatorname{pd} S(i) = \begin{cases} 2n-1, & \text{if } i = 1; \\ 1, & \text{if } 2 \le i \le 2n-1; \\ 0, & \text{if } i = 2n; \\ 2j+1, & \text{if } i = 2n+n-j \text{ for } 0 \le j < n-1; \\ 2n-2, & \text{if } i = 2n+1; \end{cases}$$

$$\operatorname{id} S(i) = \begin{cases} 0, & \text{if } i = 1; \\ 1, & \text{if } 2 \le i \le 2n; \\ 2n-1, & \text{if } i = 2n+1; \\ 2j-2, & \text{if } i = 2n+j \text{ for } 2 \le j \le n. \end{cases}$$

Hence $\mathrm{LL}(\Lambda) = 2n$ and $\operatorname{gl.dim} \Lambda = 2n-1$. So $\mathcal{S}^{<\infty} = \{$all simple modules in $\operatorname{mod}\Lambda\}$. Let $\mathcal{S} := \{S(i) \mid 2 \le i \le 2n\} (\subseteq \mathcal{S}^{<\infty})$ and $\mathcal{S}'$ be all the others simple modules in $\operatorname{mod}\Lambda$. Then $\operatorname{pd}\mathcal{S} = 1 = \operatorname{id}\mathcal{S}$ and $\mathcal{S}' = \{S(i) \mid i = 1$ or $2n+1 \le$

$i \leq 2n+n\}$. Because $\Lambda = \oplus_{i=1}^{3n} P(i)$, we have $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(P(i)) \mid 1 \leq i \leq 3n\}$ by [4, Lemma 3.4(a)].

In order to compute $\ell\ell^{t_{\mathcal{S}}}(P(i))$, we need to find the smallest non-negative integer j such that $t_{\mathcal{S}} F_{t_{\mathcal{S}}}^{j}(P(i)) = 0$. Since $\operatorname{top} P(1) = S(1) \in \operatorname{add} \mathcal{S}'$, we have $t_{\mathcal{S}}(P(1)) = P(1)$ by [4, Proposition 5.9(a)]. Thus

$$F_{t_{\mathcal{S}}}(P(1)) = \operatorname{rad} t_{\mathcal{S}}(P(1)) = \operatorname{rad}(P(1)) = S(2n+1) \oplus P(2).$$

Since $\operatorname{top} S(2n+1) = S(2n+1) \in \operatorname{add} \mathcal{S}'$, we have $t_{\mathcal{S}}(S(2n+1)) = S(2n+1)$ by [4, Proposition 5.9(a)]. Since $P(2) \in \mathfrak{F}(\mathcal{S})$, we have $t_{\mathcal{S}}(P(2)) = 0$ by [4, Proposition 5.3]. So

$$t_{\mathcal{S}} F_{t_{\mathcal{S}}}(P(1)) = t_{\mathcal{S}}(S(2n+1) \oplus P(2)) = S(2n+1).$$

It follows that

$$F_{t_{\mathcal{S}}}^{2}(P(1)) = \operatorname{rad} t_{\mathcal{S}} F_{t_{\mathcal{S}}}(P(1)) = \operatorname{rad}(S(n+1)) = 0$$

and $t_{\mathcal{S}} F_{t_{\mathcal{S}}}^{2}(P(1)) = 0$, which implies $\ell\ell^{t_{\mathcal{S}}}(P(1)) = 2$. Similarly, we have

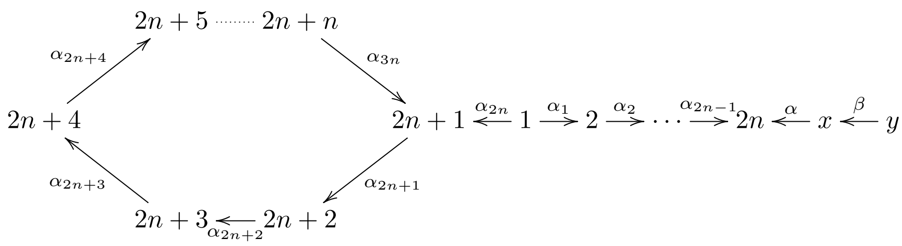
$$\ell\ell^{t_{\mathcal{S}}}(P(i)) = \begin{cases} 0, & \text{if } 2 \leq i \leq 2n; \\ 2, & \text{if } i = 1; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n; \end{cases}$$

$$\ell\ell^{t_{\mathcal{S}}}(I(i)) = \begin{cases} 1, & \text{if } 1 \leq i \leq 2n; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n; \end{cases}$$

Consequently, we conclude that $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(P(i)) \mid 1 \leq i \leq 3n\} = n+1 = \ell\ell^{t_{\mathcal{S}}}(D\Lambda)$. $\dim \operatorname{mod} \Lambda \leq \operatorname{id} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = n+2$.

The following example shows that $\dim \operatorname{mod} \Lambda \leq \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = n+2 < \operatorname{id} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda) = n+3$.

Example 5.2. Consider the bound quiver algebra $\Lambda = kQ/I$, where k is a field and Q is given by



and I is generated by $\{\beta\alpha; \alpha_{2n}\alpha_{2n+1}; \omega_i \text{ for } 1 \leq i \leq n-1\}$ where $\omega_1 = \alpha_{2n+1}\alpha_{2n+2}\cdots\alpha_{2n+n}$; $\omega_i = \alpha_{2n+i}\alpha_{2n+i+1}\alpha_{2n+i+2}\cdots\alpha_{2n+i-1}\alpha_{2n+i}$ for $2 \leq i \leq$

$n-1$. Then the indecomposable projective Λ-modules are

$$P(1)=\begin{array}{ccc} & & 1\\ & \swarrow & \downarrow\\ 2n+1 & & 2\\ & & \downarrow\\ & & 3\\ & & \downarrow\\ & & \vdots\\ & & \downarrow\\ & & 2n \end{array} \quad P(j)=\begin{array}{c} j\\ \downarrow\\ j+1\\ \downarrow\\ \vdots\\ \downarrow\\ 2n \end{array} \quad P(2n+1)=\begin{array}{c} 2n+1\\ \downarrow\\ 2n+2\\ \downarrow\\ 2n+3\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+n \end{array} \quad P(2n+i)=\begin{array}{c} 2n+i\\ \downarrow\\ 2n+i+1\\ \downarrow\\ 2n+i+2\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+i-1\\ \downarrow\\ 2n+i \end{array}$$

$$P(x)=\begin{array}{c} x\\ \downarrow\\ 2n \end{array} \qquad P(y)=\begin{array}{c} y\\ \downarrow\\ x \end{array}$$

where $2\leq j\leq 2n$, $2\leq i\leq n$. Then the indecomposable injective Λ-modules are

$$I(j)=\begin{array}{c} 1\\ \downarrow\\ 2\\ \downarrow\\ 3\\ \downarrow\\ \vdots\\ \downarrow\\ j \end{array} \quad I(2n)=\begin{array}{ccc} 1 & & \\ \downarrow & & \\ 2 & & \\ \downarrow & & \\ 3 & & \\ \downarrow & & \\ \vdots & & x\\ \downarrow & \swarrow & \\ 2n & & \end{array} \quad I(2n+1)=\begin{array}{ccc} 2n+2 & & \\ \downarrow & & \\ 2n+3 & & \\ \downarrow & & \\ 2n+4 & & \\ \downarrow & & \\ \vdots & & 1\\ \downarrow & \swarrow & \\ 2n+1 & & \end{array} \quad I(2n+i)=\begin{array}{c} 2n+i\\ \downarrow\\ 2n+i+1\\ \downarrow\\ 2n+i+2\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+i-1\\ \downarrow\\ 2n+i \end{array}$$

$$I(x)=\begin{array}{c} y\\ \downarrow\\ x \end{array} \qquad I(y)=y$$

where $1\leq j\leq 2n-1$, $2\leq i\leq n$.

We have

$$\operatorname{pd} S(i)=\begin{cases} 2n-1, & \text{if } i=1;\\ 1, & \text{if } 2\leq i\leq 2n-1 \text{ or } i=x;\\ 0, & \text{if } i=2n;\\ 2, & \text{if } i=y;\\ 2j+1, & \text{if } i=2n+n-j \text{ for } 0\leq j<n-1;\\ 2n-2, & \text{if } i=2n+1; \end{cases}$$

$$\operatorname{id} S(i) = \begin{cases} 0, & \text{if } i = 1 \text{ or } y; \\ 1, & \text{if } 2 \leq i \leq 2n-1 \text{ or } i = x; \\ 2, & \text{if } i = 2n; \\ 2n-1, & \text{if } i = 2n+1; \\ 2j-2, & \text{if } i = 2n+j \text{ for } 2 \leq j \leq n. \end{cases}$$

Hence $\mathrm{LL}(\Lambda) = 2n$ and $\operatorname{gl.dim} \Lambda = 2n-1$. So $\mathcal{S}^{<\infty} = \{\text{all simple modules in } \operatorname{mod} \Lambda\}$. Let $\mathcal{S} := \{S(i) \mid 2 \leq i \leq 2n\} (\subseteq \mathcal{S}^{<\infty})$ and $\mathcal{S}'$ be all the others simple modules in $\operatorname{mod} \Lambda$. Then $\operatorname{pd} \mathcal{S} = 1$, $\operatorname{id} \mathcal{S} = 2$ and $\mathcal{S}' = \{S(i) \mid i = 1;\ x;\ y;\ 2n+1 \leq i \leq 2n+n\}$.

Similarly as above, we have

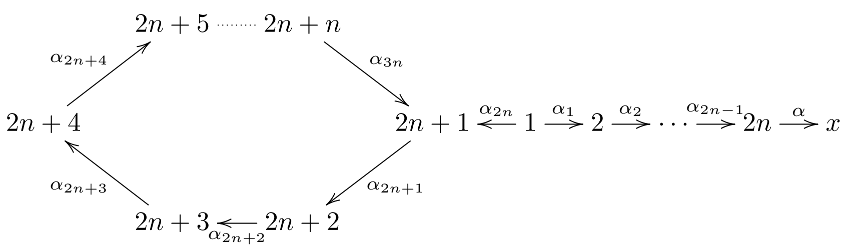
$$\ell\ell^{t_{\mathcal{S}}}(P(i)) = \begin{cases} 0, & \text{if } 2 \leq i \leq 2n; \\ 1, & \text{if } i = x; \\ 2, & \text{if } i = 1, \text{ or } i = y; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n; \end{cases}$$

$$\ell\ell^{t_{\mathcal{S}}}(I(i)) = \begin{cases} 1, & \text{if } 1 \leq i \leq 2n, \text{ or } i = y; \\ 2, & \text{if } i = x; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n. \end{cases}$$

Consequently, we conclude that $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(P(i)) \mid 1 \leq i \leq 3n\} = n+1 = \ell\ell^{t_{\mathcal{S}}}(D\,\Lambda)$. $\dim \operatorname{mod} \Lambda \leq \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = n+2 < \operatorname{id} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = n+3$.

The following example shows that $\dim \operatorname{mod} \Lambda \leq \operatorname{id} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(D\Lambda) = n+2 < \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = n+3$.

Example 5.3. Consider the bound quiver algebra $\Lambda = kQ/I$, where k is a field and Q is given by



and I is generated by $\{\alpha_{2n-1}\alpha; \alpha_{2n}\alpha_{2n+1}; \omega_i \text{ for } 1 \leq i \leq n-1\}$ where $\omega_1 = \alpha_{2n+1}\alpha_{2n+2}\cdots\alpha_{2n+n}$; $\omega_i = \alpha_{2n+i}\alpha_{2n+i+1}\alpha_{2n+i+2}\cdots\alpha_{2n+i-1}\alpha_{2n+i}$ for $2 \leq i \leq$

$n-1$. Then the indecomposable projective Λ-modules are

$$P(1)=\begin{array}{cc} & 1\\ \swarrow & \downarrow\\ 2n+1 & 2\\ & \downarrow\\ & 3\\ & \downarrow\\ & \vdots\\ & \downarrow\\ & 2n\end{array}\qquad P(j)=\begin{array}{c} j\\ \downarrow\\ j+1\\ \downarrow\\ \vdots\\ \downarrow\\ 2n\end{array}\qquad P(2n+1)=\begin{array}{c} 2n+1\\ \downarrow\\ 2n+2\\ \downarrow\\ 2n+3\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+n\end{array}\qquad P(2n+i)=\begin{array}{c} 2n+i\\ \downarrow\\ 2n+i+1\\ \downarrow\\ 2n+i+2\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+i-1\\ \downarrow\\ 2n+i\end{array}$$

$$P(2n)=\begin{array}{c} 2n\\ \downarrow\\ x\end{array}\qquad P(x)=x$$

where $2\le j\le 2n-1$, $2\le i\le n$. Then the indecomposable injective Λ-modules are

$$I(j)=\begin{array}{c} 1\\ \downarrow\\ 2\\ \downarrow\\ 3\\ \downarrow\\ \vdots\\ \downarrow\\ j\end{array}\qquad I(2n+1)=\begin{array}{cc} 2n+2 & \\ \downarrow & \\ 2n+3 & \\ \downarrow & \\ 2n+4 & \\ \downarrow & \\ \vdots & 1\\ \downarrow & \swarrow\\ 2n+1 & \end{array}\qquad I(2n+i)=\begin{array}{c} 2n+i\\ \downarrow\\ 2n+i+1\\ \downarrow\\ 2n+i+2\\ \downarrow\\ \vdots\\ \downarrow\\ 2n+i-1\\ \downarrow\\ 2n+i\end{array}\qquad I(x)=\begin{array}{c} 2n\\ \downarrow\\ x\end{array}$$

where $1\le j\le 2n$, $2\le i\le n$.

We have

$$\operatorname{pd}S(i)=\begin{cases} 2n-1, & \text{if } i=1;\\ 1, & \text{if } 2\le i<2n-1 \text{ or } i=2n;\\ 0, & \text{if } i=x;\\ 2, & \text{if } i=2n-1;\\ 2j+1, & \text{if } i=2n+n-j \text{ for } 0\le j<n-1;\\ 2n-2, & \text{if } i=2n+1;\end{cases}$$

$$\operatorname{id}S(i)=\begin{cases} 0, & \text{if } i=1\ ;\\ 1, & \text{if } 2\le i\le 2n;\\ 2, & \text{if } i=x;\\ 2n-1, & \text{if } i=2n+1;\\ 2j-2, & \text{if } i=2n+j \text{ for } 2\le j\le n.\end{cases}$$

Hence $\mathrm{LL}(\Lambda) = 2n$ and $\mathrm{gl.dim}\,\Lambda = 2n-1$. So $\mathcal{S}^{<\infty} = \{$all simple modules in $\mathrm{mod}\,\Lambda\}$. Let $\mathcal{S} := \{S(i) \,|\, 2 \leq i \leq 2n\} (\subseteq \mathcal{S}^{<\infty})$ and $\mathcal{S}'$ be all the others simple modules in $\mathrm{mod}\,\Lambda$. Then $\mathrm{pd}\,\mathcal{S} = 2$, $\mathrm{id}\,\mathcal{S} = 1$ and $\mathcal{S}' = \{S(i) \,|\, i = 1;\ x;\ 2n+1 \leq i \leq 2n+n\}$.

Similarly as above, we have

$$\ell\ell^{t_\mathcal{S}}(P(i)) = \begin{cases} 0, & \text{if } 2 \leq i < 2n; \\ 1, & \text{if } i = x; \text{ or } i = 2n; \\ 2, & \text{if } i = 1; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n; \end{cases}$$

$$\ell\ell^{t_\mathcal{S}}(I(i)) = \begin{cases} 1, & \text{if } 1 \leq i \leq 2n; \text{ or } i = x; \\ n, & \text{if } i = 2n+1; \\ n+1, & \text{if } 2n+2 \leq i \leq 2n+n. \end{cases}$$

Consequently, we conclude that $\ell\ell^{t_\mathcal{S}}(\Lambda) = \max\{\ell\ell^{t_\mathcal{S}}(P(i)) \,|\, 1 \leq i \leq 3n\} = n+1 = \ell\ell^{t_\mathcal{S}}(D\Lambda)$, $\dim \mathrm{mod}\,\Lambda \leq \mathrm{id}\,\mathcal{S} + \ell\ell^{t_\mathcal{S}}(\Lambda) = n+2 < \mathrm{pd}\,\mathcal{S} + \ell\ell^{t_\mathcal{S}}(\Lambda) = n+3$.

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