EXISTENCE AND MULTIPlicity OF Solutions FOR KIRCHHOFF-SCHRÖDINGER-POISSON SYSTEM WITH CONCAVE AND CONVEX NONLINEARITIES

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Abstract. This paper is concerned with the following Kirchhoff-Schrödinger-Poisson system
\[
\begin{aligned}
&-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V(x)u + \mu \phi u = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\
&-\Delta \phi = \mu |u|^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
where \(a > 0, \ b, \ \mu \geq 0, \ p \in (1, 2), \ q \in [4, 6)\) and \(\lambda > 0\) is a parameter. Under some suitable assumptions on \(V(x), \ f(x)\) and \(g(x)\), we prove that the above system has at least two different nontrivial solutions via the Ekeland’s variational principle and the Mountain Pass Theorem in critical point theory. Some recent results from the literature are improved and extended.

1. Introduction

This paper deals with the existence and multiplicity of nontrivial solutions for the following Kirchhoff-Schrödinger-Poisson system
\[
\begin{aligned}
&-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V(x)u + \mu \phi u = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\
&-\Delta \phi = \mu |u|^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
where \(a > 0, \ b, \ \mu \geq 0, \ p \in (1, 2), \ q \in [4, 6)\) and \(\lambda > 0\) is a parameter.

When \(\mu = 0\), then system (1.1) becomes the following Kirchhoff-type equations
\[
\begin{aligned}
&-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]
Eq. (1.2) is related to the stationary analogue of the Kirchhoff equation

\begin{equation}
\begin{aligned}
\dot{u} - (a + b \int_{\mathbb{R}^3} |\nabla_x u|^2 \, dx) \nabla_x u = \lambda f(x)|u|^{p-2} u + g(x)|u|^{q-2} u,
\end{aligned}
\end{equation}

which was proposed by Kirchhoff [18] as an extension of the classical D’Alembert wave equation

\begin{equation}
\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0
\end{aligned}
\end{equation}

for free vibrations of elastic strings. Kirchhoff’s modes takes into account the changes in length of the string produced by transverse vibrations. Here, $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension.

In [1], it was pointed out that problem (1.3) models several physical systems, where $u$ describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems. A parabolic version of problem (1.1) can, in theory, be used to describe the growth and the movement of a particular species. The movement, modeled by the integral term, is assumed to be independent on the energy of the entire system with $u$ being its population density. Alternatively, the movement of a particular species may be subject to the total population with the domain (for instance, the spreading of bacteria), which gives rise to equations of the type $u_t - a \left( \int_{\mathbb{R}^3} u \, dx \right) \Delta u = h$.

Some early classical studies of Kirchhoff equations were those of Pohozaev [27]. While, problem (1.2) received large attention only after Lions [19] proposed an abstract framework for the problem. Some interesting results for problem (1.2) can be found in [3, 7, 14] and the references therein.

Recently, by use of variational methods, lots of important results on the existence and multiplicity of solutions for elliptic equations have been obtained, see [8–13, 21, 23, 24, 26, 31–33] and the references therein. For instance, Mao et al. [24] studied the existence of sign-changing solutions for problem (1.2) by using the minimax methods and the invariant sets of descent flow when the nonlinearity satisfies the asymptotically 3-linear growth condition. Later, in [23], when the nonlinearity satisfies super-3-linear growth condition, they also obtained the existence of sign-changing solutions for problem (1.2). Very recently, by combining the constraint methods and the quantitative deformation lemma, Tang and Cheng [33] investigated the existence of a ground state sign-changing solution and proved that its energy is strictly larger than twice that of the ground state solution of Nehari-type. Moreover, the convergence property of the ground state sign-changing solution was also obtained by them.

When $a = 1$ and $b = 0$, problem (1.1) reduces to the following Schrödinger–Poisson system

\begin{equation}
\begin{aligned}
\begin{cases}
-\Delta u + V(x)u + \mu \phi u = \lambda f(x)|u|^{p-2} u + g(x)|u|^{q-2} u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi = \mu |u|^2, & \text{in } \mathbb{R}^3.
\end{cases}
\end{aligned}
\end{equation}
Such a system, also known as the Schrödinger-Maxwell system, arises in an interesting physical context. Indeed, according to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger equations and the Poisson equations (for more details on the physical aspects, we refer the reader to [6] and the references therein). In particular, if we are looking for electrostatic type solutions, we just need to solve problem (1.5).

In recent years, problem (1.5) has been widely studied via variational methods under various hypotheses on the potential function and the nonlinearity, see [4, 16, 20, 22, 28–30, 35] and the references therein. For example, Sun et al. [30] proved the existence of positive solutions for problem (1.5) when the nonlinearity has growth at most linear for $\mu$ small, allowing the potential $V(x)$ to vanish at infinity. In addition, they obtained the nonexistence of a nontrivial positive solution for $\mu \geq \frac{1}{4}$. When $p \in (2, 6)$ and $g(x) \equiv 0$, Liu et al. [22] investigated the existence and asymptotic properties of the ground solutions for problem (1.5). Moreover, some non-existence results were proved by them. Xu and Chen [35] obtained the existence of multiple negative energy solutions for problem (1.5) by using the variant fountain theorem established by Zou [37]. In [28], when $V(x) = f(x) \equiv 1, g(x) = 0$ and $\lambda \equiv 1$, the existence and nonexistence results on positive radial solutions for problem (1.5) were obtained, depending on the parameters $p$ and $\mu$. It turns out $p = 3$ is a critical value for the existence of solutions.

In [36], Zhao et al. considered the following nonlinear Kirchhoff-Schrödinger-Poisson system

\[(1.6) \begin{cases} (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \left[ -\Delta u + V(x)u \right] + \lambda(x)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \lambda(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \]

where $a > 0$ and $b, l \geq 0$. When $f(x,u)$ has sublinear growth in $u$ and $V(x)$ does not have a positive lower bound, they obtained the existence of infinitely many solutions for the above system via the symmetric mountain pass theorem established by Kajikiya (see [17]).

Motivated by the above facts, in this paper, our aim is to study the existence and multiplicity of solutions for Kirchhoff-Schrödinger-Poisson system. Our tools are the Mountain Pass Theorem and the Ekeland's variational principle.

We assume that functions $V(x), f(x)$ and $g(x)$ satisfy the following hypotheses.

\[ (V_1) \ V \in C(\mathbb{R}^3, \mathbb{R}) \ \text{and } V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0; \]

\[ (F) \ f \in L_{\text{loc}}^{\frac{12p}{12p-3}}(\mathbb{R}^3, \mathbb{R}), f(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^3 \ \text{and } f(x) \neq 0; \]

\[ (G) \ g \in L^{\infty}(\mathbb{R}^3, \mathbb{R}), g(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^3 \ \text{and } g(x) \neq 0. \]

Now, we state our main results.

**Theorem 1.1.** Suppose that conditions $(V_1), (F)$ and $(G)$ hold, if $V(x), f(x)$ and $g(x)$ are radial functions, then there exists $\Lambda_1 > 0$ such that problem (1.1)
possesses at least two different nontrivial solutions whenever $0 < \lambda < \Lambda_1$. Moreover, $u$ is symmetric.

Remark 1.1. To the best of our knowledge, it seems that Theorem 1.1 is the first result about the existence of multiple solutions for the Kirchhoff-Schrödinger-Poisson equations with concave and convex nonlinearities on $\mathbb{R}^3$.

Next, we study the existence of multiple nontrivial solutions for the following Kirchhoff-Schrödinger-Poisson system

\begin{equation}
\begin{aligned}
- (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x) u + \mu \phi u = \lambda f(x) |u|^{p-2} u + h(x, u), & \quad \text{in } \mathbb{R}^3, \\
- \Delta \phi = \mu |u|^2, & \quad \text{in } \mathbb{R}^3.
\end{aligned}
\end{equation}

Precisely, we make the following hypotheses.

\begin{itemize}
    \item [(V_2)] For every $M > 0$, $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$, where $\text{meas}$ denotes the Lebesgue measure in $\mathbb{R}^3$;
    \item [(H_1)] $h \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $\alpha \in (2, 6)$ and $c_1 > 0$ such that $|h(x, u)| \leq c_1 (1 + |u|^{\alpha-1})$, $\forall x \in \mathbb{R}^3$ and $u \in \mathbb{R}$;
    \item [(H_2)] $\frac{h(x, u)}{u} \to 0$ as $u \to 0$ uniformly for $x \in \mathbb{R}^3$;
    \item [(H_3)] there exist $\gamma > 4$ and $R_0 > 0$ such that
    \begin{equation*}
    \gamma H(x, u) := \gamma \int_0^u h(x, s) \, ds \leq uh(x, u)
    \end{equation*}
    for every $x \in \mathbb{R}^3$ and all $|u| \geq R_0$;
    \item [(H_4)] $\inf_{x \in \mathbb{R}^3, |u| = 1} H(x, u) > 0$.
\end{itemize}

We can now state the second result.

**Theorem 1.2.** Suppose that conditions (V_1), (V_2), (F) and (H_1)-(H_4) hold, then there exists $\Lambda_2 > 0$ such that problem (1.7) possesses at least two different nontrivial solutions whenever $0 < \lambda < \Lambda_2$.

Remark 1.2. (i) The condition (V_2), which implies the compactness of embedding from the working space to $L^r(\mathbb{R}^3), 2 \leq r < 6$, and contains the coercivity: $V(x) \to \infty$ as $|x| \to \infty$, was first introduced by Bartsch and Wang [5] to overcome the lack of compactness. The conditions (H_2)-(H_3) were firstly introduced by Ambrosetti and Rabinowitz in [2], and the condition (H_4) was first introduced by Mugnai in [25].

(ii) Compared with the results of Theorem 1.1, the existence of multiple nonradially symmetric solutions for problem (1.7) can be obtained in Theorem 1.2 since the potential $V(x)$ may be not radially symmetric.

This paper is organized as follows. In Section 2, by using the Mountain Pass theorem and the Ekeland’s variational principle, we obtain the existence of two nontrivial radial solutions for problem (1.1). In Section 3, the existence of two nontrivial solutions for problem (1.7) is proved.
Notations.
• $|\cdot|_r = \left( \int_{\mathbb{R}^3} |u|^r \, dx \right)^{\frac{1}{r}}$, $1 \leq r \leq \infty$.
• $C$ denotes a positive constant, which may vary from line to line.
• $S_p$ denotes the best Sobolev constant:
  \[ S_p = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) \, dx}{(\int_{\mathbb{R}^3} |u|^p \, dx)^{\frac{2}{p}}}, \quad p \in [2, 6). \]

2. Proof of Theorem 1.1

Before giving the proofs of our main results, we firstly give some preliminary results.

Let
\[ H^1(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) | \nabla u \in L^2(\mathbb{R}^3) \} \]
be equipped with the inner product and the norm
\[ \langle u, v \rangle_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) \, dx, \quad \|u\|_{H^1} = \langle u, v \rangle_{H^1}^{\frac{1}{2}}. \]

Define
\[ D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) | \nabla u \in L^2(\mathbb{R}^3) \} \]
with the inner product and the norm
\[ \langle u, v \rangle_{D^{1,2}} = \int_{\mathbb{R}^3} \nabla u \nabla v \, dx, \quad \|u\|_{D^{1,2}} = \langle u, v \rangle_{D^{1,2}}^{\frac{1}{2}}. \]

For any $u \in H^1(\mathbb{R}^3)$, from the condition $(F)$, $\frac{6}{p} \in (3, 6)$, $\frac{12p}{6-p} \in \left( \frac{5}{2}, 6 \right)$ and the Hölder inequality, we have
\[
\int_{\mathbb{R}^3} f(x)|u|^p \, dx = \int_{\mathbb{R}^3} f(x)|u|^{p-1}|u| \, dx \\
\leq \left( \int_{\mathbb{R}^3} (f(x)|u|^{p-1})^{\frac{6}{6-p}} \, dx \right)^{\frac{6-p}{6}} \left( \int_{\mathbb{R}^3} |u|^\frac{6}{6-p} \, dx \right)^{\frac{6}{6-p}} \\
\leq \left( \int_{\mathbb{R}^3} f(x)|u|^{p-1} \, dx \right)^{\frac{6}{6-p}} \left( \int_{\mathbb{R}^3} |u|^{\frac{12p}{6-p}} \, dx \right)^{\frac{6-p}{12p}} |u|^{\frac{6}{6-p}} \\
= \left| f \right|^{\frac{12p}{1+p(p-6)}} |u|^{p-1} |u|^{\frac{6}{6-p}} \\
\leq |f|^{\frac{12p}{1+p(p-6)}} S^{\frac{1}{12p}} S^{\frac{6}{p}} \frac{\mu}{p} \|u\|^p_{H^1} := C(f, p) \|u\|^p_{H^1}.
\]

Consider the linear functional $Lu$ defined in $D^{1,2}(\mathbb{R}^3)$ by
\[ Lu(v) = \mu \int_{\mathbb{R}^3} u^2 v \, dx. \]
One can check that the functional $Lu$ is continuous in $D^{1,2}(\mathbb{R}^3)$. Therefore, the Hölder inequality and the Sobolev inequality imply that
\begin{equation}
|Lu(v)| \leq \mu |u^2|_4^{|v|_6} = \mu |u^2|_4^{|v|_6} \leq \mu C |u^2|_4^{|v|_{D^{1,2}}} 
\leq \mu C |u|_{H^1}^2 |v|_{D^{1,2}}.
\end{equation}

Hence, the Lax-Milgram Theorem implies that for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that
\begin{equation}
\mu \int_{\mathbb{R}^3} u^2 \nu dx = \int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx, \forall v \in D^{1,2}(\mathbb{R}^3).
\end{equation}

Moreover, $\phi_u$ has the following integral expression
\begin{equation}
\phi_u(x) = \frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \geq 0.
\end{equation}

Using integration by parts, we get
\begin{equation}
\int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx = - \int_{\mathbb{R}^3} u \Delta \phi_u dx, \forall v \in D^{1,2}(\mathbb{R}^3),
\end{equation}

and $-\Delta \phi_u = \mu u^2$ in a weak sense.

In addition, by (2.2) and (2.3), we easily obtain that
\begin{equation}
||\phi_u||_{D^{1,2}} \leq \mu C ||u||_{H^1} ||\phi_u||_{D^{1,2}},
\end{equation}

that is
\begin{equation}
||\phi_u||_{D^{1,2}} \leq \mu C ||u||_{H^1}^2,
\end{equation}

and combine (2.2), one has
\begin{equation}
\frac{\mu^2}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dy dx = \mu \int_{\mathbb{R}^3} u^2 \phi_u(x) dx \leq \mu^2 C ||u||_{H^1}^2.
\end{equation}

We define the energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ as follows:
\begin{equation}
I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 
+ \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} f(x)|u|^p dx - \frac{1}{4} \int_{\mathbb{R}^3} g(x)|u|^q dx.
\end{equation}

Then it follows from (2.1) and (2.6) that $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and that
\begin{equation}
\langle I'(u), v \rangle = \int_{\mathbb{R}^3} a \nabla u \nabla v dx + \int_{\mathbb{R}^3} V(x)uv dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla v dx 
+ \mu \int_{\mathbb{R}^3} \phi_u uv dx - \lambda \int_{\mathbb{R}^3} f(x)|u|^{p-2} uv dx - \int_{\mathbb{R}^3} g(x)|u|^{q-2} uv dx.
\end{equation}

Therefore, if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I$, then $u$ is a solution of problem (1.1).

Since problem (1.1) is set on $\mathbb{R}^3$, so the Sobolev embedding
\begin{equation}
H^1(\mathbb{R}^3) \hookrightarrow L^{r}(\mathbb{R}^3), \ r \in [2,6)
\end{equation}
is not compact, then it is not easy to prove a minimizing sequence or a Palais-Smale sequence is strongly convergent if we look for solutions to problem (1.1) via variational methods. To overcome this difficulty, we restrict problem (1.1) in the radial function space, where the function \( u = u(r), r = |x| \). More precisely, we shall consider the functional \( I \) on the space of radial functions

\[
H^1_0(\mathbb{R}^3) := \{ u \in H^1(\mathbb{R}^3) : u = u(r), \ r = |x| \}.
\]

\( H^1_0(\mathbb{R}^3) \) is a natural constraint for \( I \), i.e., any critical point \( u \in H^1_0(\mathbb{R}^3) \) of \( I|_{H^1_0(\mathbb{R}^3)} \) is also a critical point of \( I \). Therefore, we are reduced to seek critical points of \( I|_{H^1_0(\mathbb{R}^3)} \).

**Lemma 2.1.** Suppose that conditions \((V_1)\), \((F)\) and \((G)\) hold, then there exist \( \rho_1, \alpha_1 \) and \( \Lambda_1 > 0 \) such that \( I(u)|_{\|u\|_{H^1} = \rho_1} > \alpha_1 \) for all \( \lambda \) satisfying \( \lambda \in (0, \Lambda_1) \).

**Proof.** It follows from (2.1), (2.7) and the condition \((G)\) that

\[
I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2
+ \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} f(x) |u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^3} g(x) |u|^q \, dx
\geq \min\{a, V_0\} \frac{\|u\|_{H^1}^2}{2} - \frac{\lambda C(f, p)}{p} \|u\|_{H^1}^p - \frac{S_q^{-\frac{2}{q}}}{q} |g|_{\infty} \|u\|_{H^1}^q - \frac{\lambda C(f, p)}{p}.
\]

Set

\[
l_1(t) = \frac{\min\{a, V_0\}}{2} t^{2-p} - \frac{S_q^{-\frac{2}{q}}}{q} |g|_{\infty} t^{q-p}, \quad t > 0.
\]

Then

\[
l_1'(t) = \frac{\min\{a, V_0\}}{2} (2 - p) t^{1-p} - \frac{S_q^{-\frac{2}{q}}}{q} |g|_{\infty} (q - p) t^{q-p-1} = 0, \quad t > 0,
\]

has the only solution

\[
l_1 = \left[ \frac{q(2 - p) S_q^{-\frac{2}{q}} \min\{a, V_0\}}{2(q - p) |g|_{\infty}} \right]^{\frac{1}{p-1}} > 0.
\]

Thus, we obtain that \( l_1(t_1) = \max_{t \geq 0} l_1(t) > 0 \). Choosing \( \|u\|_{H^1} := t_1 := \rho_1 \), then for every \( 0 < \lambda < \frac{\rho_1^{\frac{p}{2}}(\rho_1)}{2C(f, p)} := \Lambda_1 \), we obtain

\[
I(u)|_{\|u\|_{H^1} = \rho_1} \geq \rho_1^{\frac{p}{2}} l_1(\rho_1) - \frac{\lambda C(f, p)}{p} \geq \frac{\rho_1^{\frac{p}{2}} l_1(\rho_1)}{2} := \alpha_1.
\]

The proof is complete. □
Lemma 2.2. Suppose that conditions \((V_1)\), \((F)\) and \((G)\) hold. Then there exists a function \(e_1 \in H^1_0(\mathbb{R}^3)\) with \(\|e_1\|_{H^1} > \rho_1\) such that \(I(e_1) < 0\), where \(\rho_1\) is given by Lemma 2.1.

Proof. From (2.6), (2.7), \((F)\), \((G)\) and \(q \in (4, 6)\), we derive

\[
I(tu) = \frac{at^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{bt^2}{2} \int_{\mathbb{R}^3} V(x)u^2 \, dx + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2
\]

\[
+ \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi_n u^2 \, dx - \frac{\lambda p}{p} \int_{\mathbb{R}^3} f(x)|u|^p \, dx - \frac{t^q}{q} \int_{\mathbb{R}^3} g(x)|u|^q \, dx
\]

\[
(2.10)
\]

as \(t \to +\infty\) for \(u \in H^1_0(\mathbb{R}^3)\), \(u \neq 0\). This lemma is proved by taking \(e_1 = t_1 u\) with \(u \neq 0\) and \(t_1 > 0\) large enough. The proof is complete. \(\square\)

Lemma 2.3. Suppose that conditions \((V_1)\), \((F)\) and \((G)\) hold. If \(\{u_n\}_n \subset H^1_0(\mathbb{R}^3)\) is a bounded Palais-Smale sequence of \(I\), then \(\{u_n\}_n\) has a strongly convergent subsequence in \(H^1_0(\mathbb{R}^3)\).

Proof. Consider a sequence \(\{u_n\}_n \subset H^1_0(\mathbb{R}^3)\) satisfying

\[
I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{and} \quad \sup_n \|u_n\|_{H^1} < \infty.
\]

Then there exists \(u \in H^1_0(\mathbb{R}^3)\) such that

\[
(2.11) \quad u_n \rightharpoonup u \quad \text{in} \quad H^1_0(\mathbb{R}^3) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^r(\mathbb{R}^3) \quad \text{for} \quad r \in (2, 6).
\]

It follows from (2.11) that

\[
a_n(1) = \langle I'(u_n) - I'(u), u_n - u \rangle
\]

\[
= \left( a + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \right) \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 \, dx + \int_{\mathbb{R}^3} V(x) (u_n - u)^2 \, dx
\]

\[
+ b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) \, dx
\]

\[
(2.12) \quad + \mu \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u) (u_n - u) \, dx
\]

\[
- \lambda \int_{\mathbb{R}^3} f(x) \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) \, dx
\]

\[
- \int_{\mathbb{R}^3} g(x) \left( |u_n|^{q-2} u_n - |u|^{q-2} u \right) (u_n - u) \, dx
\]

\[
\geq \min \{a, V_0\} \|u_n - u\|_{H^1}^2 + \mu \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u) (u_n - u) \, dx
\]
From (2.11),

\[ p \in (1, 2) \]

The boundedness of \( \{ u_n \} \) implies that

\[ b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) \, dx \]

In view of (2.6) and (2.11), we obtain

\[ \lambda \int_{\mathbb{R}^3} f(x) \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) \, dx \]

\[ - \int_{\mathbb{R}^3} g(x) \left( |u_n|^{q-2} u_n - |u|^{q-2} u \right) (u_n - u) \, dx. \]

Analogously, we also get

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^p(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } H^1(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^q(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,q}(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^r(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,r}(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^\infty(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } H^{1/2}(\partial \Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1/2,1}(\partial \Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^s(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,s}(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^t(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,t}(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^\infty(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } H^{1/2}(\partial \Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1/2,1}(\partial \Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^s(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,s}(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } L^t(\Omega) \]

\[ \phi_n u_n \rightarrow \phi u \quad \text{in } W^{1,t}(\Omega) \]
Hence, we have
\[ \int_{\mathbb{R}^3} f(x) \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) (u_n - u) \, dx \]
(2.15)
\[ = \int_{\mathbb{R}^3} f(x)|u_n|^{p-2}u_n (u_n - u) \, dx - \int_{\mathbb{R}^3} f(x)|u|^{p-2}u (u_n - u) \, dx \]
\[ \to 0 \text{ as } n \to \infty. \]

On the other hand, by (2.11) and the condition (G), it is easy to verify that
\[ \int_{\mathbb{R}^3} g(x) \left( |u_n|^{q-2}u_n - |u|^{q-2}u \right) (u_n - u) \, dx \to 0 \text{ as } n \to \infty. \]
(2.16)

Then it follows from (2.12)-(2.16) that
\[ ||u_n - u||_{H^1} \to 0 \text{ as } n \to \infty. \]
The proof is complete. □

Proof of Theorem 1.1. The proof of this theorem is divided into two steps.

Step 1. There exists a function \( u_1 \in H^1_{r}(\mathbb{R}^3) \) such that \( I'(u_1) = 0 \) and \( I(u_1) < 0 \).

Since \( f \in L^{\frac{12p}{12p+6(6-p)}}(\mathbb{R}^3, \mathbb{R}) \) and \( f \neq 0 \), we can choose a function \( \varphi \in H^1_{r}(\mathbb{R}^3) \) such that
\[ \int_{\mathbb{R}^3} f(x)|\varphi|^p \, dx > 0. \]
Then by (2.10), (F), (G) and \( p \in (1, 2) \), we derive
\[ I(t\varphi) \leq \frac{t^2}{2} \int_{\mathbb{R}^3} \left( |\nabla \varphi|^2 + V(x)\varphi^2 \right) \, dx + \frac{(b+C)t^4}{4} \| \varphi \|_{H^1}^4 - \frac{t^p}{p} \int_{\mathbb{R}^3} f(x)|\varphi|^p \, dx \]
\[ < 0 \]
for \( t > 0 \) small enough. Therefore, we obtain
\[ c_1 = \inf \{ I(u) : u \in \overline{B}_{\rho_1} \} < 0, \]
where \( \rho_1 > 0 \) is given by Lemma 2.1, \( B_{\rho_1} = \{ u \in H^1_{r}(\mathbb{R}^3) : \| u \|_{H^1} < \rho_1 \} \). Then by the Ekeland’s variational principle [15], there exists a sequence \( \{ u_n \} \subset \overline{B}_{\rho} \) such that
\[ c_1 \leq I(u_n) < c_1 + \frac{1}{n}, \]
and
\[ I(w) \geq I(u_n) - \frac{1}{n} \| w - u_n \|_{H^1} \]
for all \( w \in \overline{B}_{\rho_1} \). Obviously, it follows from Lemma 2.1 that \( u_n \in B_{\rho_1} \) for \( n \) large enough. Hence, for any \( \psi \in H^1_{r}(\mathbb{R}^3) \) with \( \| \psi \|_{H^1} = 1 \), we can choose \( t > 0 \) such that \( u_n + t\psi \in \overline{B}_{\rho_1} \) for \( n \) large enough. Thus, we obtain
\[ \frac{I(u_n + t\psi) - I(u_n)}{t} \geq -\frac{1}{n}. \]
Let \( t \to 0 \), then we have
\[ \langle I'(u_n), \psi \rangle \geq -\frac{1}{n}. \]
(2.17)
Replacing $\psi$ by $-\psi$ in (2.17), then we get

$$\langle I'(u_n), -\psi \rangle \geq -\frac{1}{n},$$

which implies that

$$\langle I'(u_n), \psi \rangle \leq \frac{1}{n}. \tag{2.18}$$

Therefore, it follows from (2.17) and (2.18) that

$$\left| \langle I'(u_n), \psi \rangle \right| \leq \frac{1}{n},$$

showing that $I'(u_n) \to 0$ as $n \to \infty$. Hence, we can conclude that $\{u_n\}$ is a bounded $(PS)\_c$ sequence of $I$. Then it follows from Lemma 2.3 that there exists a function $u_1 \in H^1_\gamma(\mathbb{R}^3)$ such that $I'(u_1) = 0$ and $I(u_1) < 0$.

**Step 2.** There exists a function $\tilde{u}_1 \in H^1_\gamma(\mathbb{R}^3)$ such that $I'(\tilde{u}_1) = 0$ and $I(\tilde{u}_1) > 0$.

By Lemma 2.1, Lemma 2.2 and the Mountain Pass Theorem (see [34]), there exists a sequence $\{u_n\} \subset H^1_\gamma(\mathbb{R}^3)$ such that

$$I(u_n) \to \tilde{c}_1 = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) \geq \alpha_1 > 0,$$

where

$$\Gamma = \{ \gamma \in C([0, 1], H^1_\gamma(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e_1 \}.$$

From Lemma 2.3, we only need to check that $\{u_n\}$ is bounded in $H^1_\gamma(\mathbb{R}^3)$. It follows from (2.1), (2.7), (V1) and $q \in (4, 6)$ that

$$q(\tilde{c}_1 + 1) + \|u_n\|_{H^1}^2 \geq qI(u_n) - \langle I'(u_n), u_n \rangle \geq \left(\frac{q}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2) \, dx + \frac{(q - 4)b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \right)^2$$

$$+ \frac{(q - 4)\mu}{4} \int_{\mathbb{R}^N} \phi_{u_n} u_n^2 \, dx + \lambda \left( 1 - \frac{q}{p} \right) \int_{\mathbb{R}^3} f(x)|u_n|^p \, dx$$

$$\geq \left( \frac{q - 2}{2} \min\{a, V_0\} \right) \|u_n\|_{H^1}^2 + \lambda \left( 1 - \frac{q}{p} \right) C(f, p)\|u_n\|_{H^1}^p,$$

for $n$ large enough. Therefore, $\{u_n\}$ is bounded in $H^1_\gamma(\mathbb{R}^3)$. The proof is complete. \qed

### 3. Proof of Theorem 1.2

Before going to the proof of Theorem 1.2, we give some useful preliminaries results.

Define the space

$$E := \left\{ u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} V(x)u^2 \, dx < +\infty \right\}$$
with the inner product and norm

\[ \langle u, v \rangle_E = \int_{\mathbb{R}^3} (a
abla u \nabla v + V(x)uv) \, dx, \quad \|u\|_E = \langle u, u \rangle_E^{1/2}. \]

Furthermore, by Lemma 3.4 in [38], we know that under the assumptions \((V_1)\) and \((V_2)\), the embedding \(E \hookrightarrow L^r(\mathbb{R}^3)\) is continuous for \(2 \leq r \leq 6\), i.e., there exists a constant \(\tau_r > 0\) such that

\[ |u|_r \leq \tau_r \|u\|_E. \tag{3.1} \]

Moreover, for every \(2 \leq r < 6\), the embedding \(E \hookrightarrow L^r(\mathbb{R}^3)\) is compact.

For any \(u \in E\), from the condition \((F)\), \((3.1)\), \(\frac{6}{p} \in (3, 6)\), \(\frac{12p}{6-p} \in (\frac{3}{2}, 6)\) and the Hölder inequality, we obtain

\[ \int_{\mathbb{R}^3} f(x)|u|^p \, dx = \int_{\mathbb{R}^3} f(x)|u|^{p-1}|u| \, dx \leq \left( \int_{\mathbb{R}^3} |f(x)| |u|^{p-1} \frac{|u|^{\frac{3}{2}}}{x} \, dx \right)^{\frac{6}{p}} \left( \int_{\mathbb{R}^3} |u|^\frac{6}{p} \, dx \right)^{\frac{p}{6}} \leq \left( \int_{\mathbb{R}^3} |f(x)| \frac{12p}{p+\frac{12p}{\frac{6}{p}-1}} \, dx \right)^{\frac{12p}{p+\frac{12p}{\frac{6}{p}-1}}} \left( \int_{\mathbb{R}^3} |u|^\frac{12p}{\frac{6}{p}-1} \, dx \right)^{\frac{\frac{6}{p}-1}{\frac{6}{p}}} \|u\|_E^p. \tag{3.2} \]

Therefore, problem (1.7) has a variational structure. We consider the functional \(I : E \rightarrow \mathbb{R}\) defined by

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^3} a|\nabla u|^2 \, dx + \int_{\mathbb{R}^3} V(x)u^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} f(x)|u|^p \, dx - \int_{\mathbb{R}^3} H(x, u) \, dx. \tag{3.3} \]

Then it follows from \((3.1)\), \((H_1)\) and \((H_2)\) that \(I \in C^1(E, \mathbb{R})\) and that

\[ \langle I'(u), v \rangle = \int_{\mathbb{R}^3} a\nabla u \nabla v \, dx + \int_{\mathbb{R}^3} V(x)uv \, dx + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \int_{\mathbb{R}^3} \nabla u \nabla v \, dx \]

\[ + \mu \int_{\mathbb{R}^3} \phi_u u v \, dx - \lambda \int_{\mathbb{R}^3} f(x)|u|^{p-2}uv \, dx - \int_{\mathbb{R}^3} h(x, u)v \, dx. \tag{3.4} \]

**Lemma 3.1.** Suppose that conditions \((V_1)\), \((V_2)\), \((F)\) and \((H_1)-(H_2)\) hold. Then there exist \(\rho_2\), \(\alpha_2\) and \(\Lambda_2 > 0\) such that \(I(u)\|u\|_E = \rho_2 \geq \alpha_2\) for all \(\lambda\) satisfying \(\lambda \in (0, \Lambda_2)\).
Proof. It follows from \((H_2)\) that for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
|h(x, u)| \leq \varepsilon |u|, \quad \forall \, x \in \mathbb{R}^3 \quad \text{and} \quad |u| \leq \delta.
\]
By \((H_1)\), for a.e. \(x \in \mathbb{R}^3\) and \(|u| \geq \delta\), we have
\[
|h(x, u)| \leq c_1 + c_1 |u|^{\alpha - 1} \leq c_1 \left( \frac{|u|}{\delta} \right)^{\alpha - 1} + c_1 |u|^{\alpha - 1} = \left( \frac{c_1}{\delta} + c_1 \right) |u|^{\alpha - 1}.
\]
It follows from (3.5) and (3.6) that
\[
|h(x, u)| \leq \varepsilon |u| + \left( \frac{c_1}{\delta} + c_1 \right) |u|^{\alpha - 1} := \varepsilon |u| + C_\varepsilon |u|^{\alpha - 1} \quad \text{for a.e.} \quad x \in \mathbb{R}^3 \quad \text{and all} \quad u \in \mathbb{R}.
\]
Therefore, by the equality \(H(x, u) = \int_0^u h(x, s)ds\), we have
\[
H(x, u) \leq \frac{\varepsilon}{2} |u|^2 + \frac{C_\varepsilon}{\alpha} |u|^\alpha \quad \text{for a.e.} \quad x \in \mathbb{R}^3 \quad \text{and all} \quad u \in \mathbb{R},
\]
where \(C_\varepsilon = \left( \frac{c_1}{\delta} + c_1 \right) / \alpha\). Then it follows from (3.1), (3.2) and (3.3) that
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2
\]
\[
+ \frac{\varepsilon}{4} \int_{\mathbb{R}^3} \phi(u) u^2 dx - \frac{\lambda}{q} \int_{\mathbb{R}^3} f(x)|u|^p dx - \int_{\mathbb{R}^3} H(x, u) dx
\]
\[
\geq \frac{1}{2} |u|^2_E - \frac{\lambda C'(f, p)}{p} |u|^p_E - \frac{\varepsilon}{2} |u|^2_E - \frac{C(\varepsilon)}{\alpha} |u|^\alpha_E
\]
\[
\geq \frac{1}{2} |u|^2_E - \frac{\varepsilon \tau_2^2}{2} |u|^2_E - \frac{\lambda C'(f, p)}{p} |u|^p_E - \frac{C_\varepsilon \tau_2^2}{\alpha} |u|^\alpha_E
\]
\[
= |u|^p_E \left[ \frac{1}{2} - \frac{\varepsilon \tau_2^2}{2} \right] |u|^2_{E^p} - \frac{\lambda C'(f, p)}{p} \frac{1}{2} - \frac{C_\varepsilon \tau_2^2}{\alpha} |u|^\alpha_{E^p}.
\]
Take \(\varepsilon = \frac{1}{2 \tau_2^2}\) and set
\[
l_2(t) = \frac{1}{4} t^{2-p} - \frac{C_\varepsilon \tau_2^2}{\alpha} t^{\alpha - p}, \quad t > 0.
\]
Then
\[
l'_2(t) = \frac{2}{4} t^{1-p} - \frac{C_\varepsilon \tau_2^2}{\alpha} (\alpha - p) t^{\alpha - p - 1}, \quad t > 0,
\]
has the only solution
\[
t_2 = \left[ \frac{\alpha(2 - p)}{4C(\varepsilon) \tau_2^2(\alpha - p)} \right]^{\frac{1}{2 - p}} > 0.
\]
Hence, we obtain that \(l_2(t_2) = \max_{t \geq 0} l_2(t) > 0\). Choosing \(|u||u|^p := t_2 := \rho_2\), then for every \(0 < \lambda < \frac{\rho_2 p(\rho_2)}{2C(f, p)} := \Lambda_2\), we obtain
\[
I(u)|_{u||u|^p := \rho_2} \geq \rho_2^p \left( l_2(\rho_2) - \frac{\lambda C'(f, p)}{p} \right) \geq \frac{\rho_2^p l_2(\rho_2)}{2} := \alpha_2.
\]
The proof is complete. □

**Lemma 3.2.** Suppose that conditions \((V_1), (V_2), (F)\) and \((H_1)-(H_4)\) hold. Then there exists a function \(e_2 \in E\) with \(\|e_2\|_E > \rho_2\) such that \(I(e_2) < 0\), where \(\rho_2\) is given in Lemma 3.1.

**Proof.** For any \((x, z) \in \mathbb{R}^3 \times \mathbb{R}\), set

\[
S(t) = t^\gamma H(x, t^{-1}z), \quad t \geq 1.
\]

Hence, for \(t \in [1, \|z\| / R_0]\) and \(\|z\| \geq R_0\), it follows from \((H_3)\) that

\[
S'(t) = t^{\gamma - 1} [\gamma H(x, t^{-1}z) - t^{-1}zh(x, t^{-1}z)] \leq 0,
\]

showing that \(S(1) \geq S\left(\frac{\|z\|}{R_0}\right)\), that is

\[
H(x, z) \geq \frac{1}{R_0^\gamma} H(x, R_0) \left|\frac{z}{|z|}\right| \geq C_{R_0} |z|^\gamma \quad \text{for } |z| \geq R_0,
\]

where \(C_{R_0} = \frac{1}{R_0^\gamma} \inf_{|z|=R_0} H(x, z) > 0\) by \((H_4)\). It follows from \((H_2)\) that there exists \(R_1 > 0\) such that

\[
\left|\frac{h(x, z)z}{z^2}\right| = \left|\frac{h(x, z)}{z}\right| \leq 1
\]

for all \(x \in \mathbb{R}^3\) and \(0 < |z| \leq R_1\). From \((H_1)\), for any \(x \in \mathbb{R}^3\) and \(R_1 \leq |z| \leq R_0\), there exists \(M > 0\) satisfying

\[
\left|\frac{h(x, z)z}{z^2}\right| \leq c_1 \left(1 + |z|^{\alpha - 1}\right) \frac{|z|}{z^2} \leq M.
\]

Therefore, it follows from (3.9) and (3.10) that

\[
|h(x, z)| \leq (M + 1) |z|^2
\]

for any \(x \in \mathbb{R}^3\) and \(|z| \leq R_0\). Since \(H(x, z) = \int_0^1 h(x, tz)zt\, dt\), then we have

\[
|H(x, z)| \leq \frac{1}{2} (M + 1) |z|^2
\]

for any \(x \in \mathbb{R}^3\) and \(|z| \leq R_0\). Then by (3.9) and (3.13), we derive

\[
H(x, z) \geq C_{R_0} |z|^\gamma - C_M |z|^2 \quad \text{for any } (x, z) \in \mathbb{R}^3 \times \mathbb{R},
\]
where $C_M = \frac{1}{2}(M+1)+C_{R_0}R_0^{-2}$. Hence, it follows from (3.3), (G) and $\gamma > 4$ that for any $u \in E$ and $u \neq 0$, there holds
\begin{align}
I(tu) &= \frac{t^2}{2} \int_{\mathbb{R}^3} a|\nabla u|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\
&\quad + \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\lambda t^4}{p} \int_{\mathbb{R}^3} f(x)|u|^p dx - \int_{\mathbb{R}^3} H(x,tu) dx \\
&\leq \frac{t^2}{2} \int_{\mathbb{R}^3} a|\nabla u|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\
&\quad + \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - C_{R_0} t^4 |u|^4 + C_M t^2 |u|^2 \to -\infty
\end{align}
(3.15)
as $t \to +\infty$. This lemma is proved by taking $e_2 = t_2 u$ with $u \neq 0$ and $t_2 > 0$ large enough. The proof is complete. \hfill \square

**Lemma 3.3.** Suppose that conditions (V₁), (V₂), (F) and (H₁)-(H₄) hold. If \( \{u_n\} \subset E \) is a bounded Palais-Smale sequence of \( I \), then \( \{u_n\} \) has a strongly convergent subsequence in \( E \).

**Proof.** Consider a sequence \( \{u_n\} \subset E \) satisfying
\[ I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{and} \quad \sup_n ||u_n||_E < \infty. \]
Then there exists $u \in E$ such that
\[ u_n \rightharpoonup u \quad \text{in} \quad E, \quad u_n \to u \quad \text{in} \quad L^r(\mathbb{R}^3), \quad r \in [2,6). \]
It follows from (3.4) that
\[ o_n(1) = \langle I'(u_n) - I'(u), u_n - u \rangle \\
= \left( a + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 dx + \int_{\mathbb{R}^3} V(x) (u_n - u)^2 dx \\
\quad + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) dx \\
\quad + \mu \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) (u_n - u) dx - \int_{\mathbb{R}^3} (h(x,u_n) - h(x,u)) (u_n - u) dx \\
\quad - \lambda \int_{\mathbb{R}^3} f(x) \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) dx \\
\geq ||u_n - u||^2_E + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) dx \\
\quad + \mu \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) (u_n - u) dx - \int_{\mathbb{R}^3} (h(x,u_n) - h(x,u)) (u_n - u) dx \\
\quad - \lambda \int_{\mathbb{R}^3} f(x) \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) dx.
\]
Similar to the proof of Lemma 2.3, we have

\begin{align}
& (3.17) \quad b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) \, dx \to 0 \text{ as } n \to \infty, \\
& (3.18) \quad \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u) (u_n - u) \, dx \to 0 \text{ as } n \to \infty, \\
& (3.19) \quad \int_{\mathbb{R}^3} f(x) \left( |u_n|^p - 2u_n - |u|^p - |u| \right) (u_n - u) \, dx \to 0 \text{ as } n \to \infty.
\end{align}

On the other hand, it follows from (3.7) and the Hölder inequality that

\[
\int_{\mathbb{R}^3} (h(x, u_n) - h(x, u)) (u_n - u) \, dx \\
\leq \int_{\mathbb{R}^3} \left[ \varepsilon (|u_n| + |u|) + C_{\varepsilon} \left( |u_n|^{\alpha} + |u|^{\alpha} \right) |u_n - u| \right] \, dx \\
\leq \varepsilon (|u_n| + |u|) |u_n - u| + C_{\varepsilon} \left( |u_n|^{\alpha} + |u|^{\alpha} \right) |u_n - u|.
\]

Since \( u_n \to u \) in \( L^r(\mathbb{R}^3) \) for any \( 2 \leq r < 6 \), then we derive

\[
(3.20) \quad \int_{\mathbb{R}^3} (h(x, u_n) - h(x, u)) (u_n - u) \, dx \to 0 \text{ as } n \to \infty.
\]

Thus it follows from (3.16)-(3.20) that \( \|u_n - u\|_E \to 0 \) as \( n \to \infty \). The proof is complete. \( \square \)

**Proof of Theorem 1.2.** The proof of this theorem is divided into two steps.

**Step 1.** There exists a function \( u_2 \in E \) such that \( I'(u_2) = 0 \) and \( I(u_2) < 0 \).

Since \( f \in L^{\frac{12}{10}}(\mathbb{R}^3, \mathbb{R}) \) and \( f \neq 0 \), we can choose a function \( \varphi \in E \) such that

\[
\int_{\mathbb{R}^3} f(x)|\varphi|^p \, dx > 0.
\]

Then by (3.15), (F) and \( p \in (1, 2) \), we obtain

\[
I(t\varphi) \leq \frac{t^2}{2} \int_{\mathbb{R}^3} a|\nabla \varphi|^2 \, dx + \frac{t^2}{2} \int_{\mathbb{R}^3} V(x)|\varphi|^2 \, dx + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx \right)^2 + \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi\varphi^2 \, dx - \frac{\lambda t^p}{p} \int_{\mathbb{R}^3} f(x)|\varphi|^p \, dx + C_M t^2 |\varphi|^2 < 0
\]

for \( t > 0 \) small enough. Hence, we derive

\[
c_2 = \inf \left\{ I(u) : u \in \overline{B}_{\rho_2} \right\} < 0,
\]

where \( \rho > 0 \) is given by Lemma 3.1, \( B_{\rho_2} = \left\{ u \in E : \|u\|_E < \rho_2 \right\} \). Then by the Ekeland’s variational principle [15], there exists a sequence \( \{u_n\} \subset B_{\rho} \) such that

\[
c_2 \leq I(u_n) < c_1 + \frac{1}{n},
\]
and

\[ I(w) \geq I(u_n) - \frac{1}{n} \|w - u_n\|_E \]

for all \( w \in B_{\rho_2} \). Thus, by a standard procedure, we can prove that \( \{u_n\} \) is a bounded Palais-Smale sequence of \( I \). It follows from Lemma 3.3 that there exists a function \( \tilde{u}_2 \in E \) such that \( I'(\tilde{u}_2) = 0 \) and \( I(\tilde{u}_2) < 0 \).

**Step 2.** There exists a function \( \tilde{u}_2 \in E \) such that \( I'(\tilde{u}_2) = 0 \) and \( I(\tilde{u}_2) > 0 \).

By Lemma 3.1, Lemma 3.2 and the Mountain Pass Theorem (see [34]), there exists a sequence \( \{u_n\} \subset E \) such that

\[ I(u_n) \to \tilde{c}_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \alpha_2 > 0, \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \]

where

\[ \Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e_2 \}. \]

From Lemma 3.3, we only need to check that \( \{u_n\} \) is bounded in \( E \). Otherwise, up to a subsequence, we may assume that \( \|u_n\|_E \to \infty \) as \( n \to \infty \). Let \( v_n = \frac{u_n}{\|u_n\|_E} \), going to a subsequence, we can assume that

\[ v_n \rightharpoonup v \quad \text{in} \quad E, \]

\[ v_n \to v \quad \text{in} \quad L^r(\mathbb{R}^3), \quad r \in [2,6), \]

\[ v_n \to v \quad \text{a.e. on} \quad \mathbb{R}^3. \]

Set \( \Omega = \{ x \in \mathbb{R}^3 : v(x) \neq 0 \} \). If \( \text{meas}(\Omega) > 0 \), then \( |u_n(x)| \to \infty \) as \( n \to \infty \) for a.e. \( x \in \Omega \). Then it follows from (3.1) and (3.14) that

\[
\int_{\mathbb{R}^3} \frac{H(x,u_n)}{\|u_n\|_E^\gamma} \, dx \geq C_{R_0} |v_n|^\gamma - C_M |v_n|^{2_\gamma},
\]

\[
\geq C_{R_0} |v_n|^\gamma - C_M \frac{\tau_0^2}{\|u_n\|_E^{2_\gamma - 2}}.
\]

Since \( \gamma > 4 \), then we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{H(x,u_n)}{\|u_n\|_E^\gamma} \, dx \geq C_{R_0} \lim_{n \to \infty} |v_n|^\gamma = C_{R_0} |x|^\gamma = C_{R_0} \int_{\Omega} |v|^\gamma \, dx > 0.
\]

It follows from (3.2) and \( p < 2 < \gamma \) that

\[
\int_{\mathbb{R}^3} \frac{f(x)|u|^p}{\|u_n\|_E^p} \, dx \leq \frac{C'(f,p)}{\|u_n\|_E^{-p}} \to 0 \quad \text{as} \quad n \to \infty.
\]

In view of (2.6), (3.3), (3.20) and (3.21), we have

\[
0 = \lim_{n \to \infty} \frac{I(u_n)}{\|u_n\|_E} = \lim_{n \to \infty} \left[ \frac{1}{2} \frac{1}{\|u_n\|_E^{\gamma - 2}} + \frac{1}{4\alpha^2} \frac{1}{\|u_n\|_E^{2_\gamma - 4}} + \frac{C}{4} \frac{1}{\|u_n\|_E^{\gamma - 4}} \right]
\]

\[
- \lim_{n \to \infty} \left[ \frac{1}{p} \int_{\mathbb{R}^3} \frac{f(x)|u|^p}{\|u_n\|_E^p} \, dx + \int_{\mathbb{R}^3} \frac{H(x,u_n)}{\|u_n\|_E} \, dx \right]
\]

\[
< 0,
\]
which is a contradiction. Hence, \( \text{meas}(\Omega) = 0 \), showing that \( v = 0 \) for a.e. \( x \in \mathbb{R}^3 \) and \( v_n \to 0 \) in \( L^r(\mathbb{R}^3) \), \( r \in [2, 6) \).

In view of (3.12) and (3.13), for every \( x \in \mathbb{R}^3 \) and \( |z| \leq R_0 \), we obtain

\[
|zh(x, z) - \gamma H(x, z)| \leq c_2 |z|^2,
\]

where \( c_2 = (\frac{\gamma}{2} + 1)(M + 1) \). Then it follows from \((H_3)\) that

\[
(3.22) \quad zh(x, z) - \gamma H(x, z) \geq -c_2 |z|^2, \quad \forall (x, z) \in \mathbb{R}^3 \times \mathbb{R}.
\]

Therefore, it follows from (3.3), (3.4) and \( \gamma > 4 \) that

\[
(3.23) \quad \gamma I(u_n) - \langle I'(u_n), u_n \rangle \geq \frac{1}{\|u_n\|_E^2} \left( \frac{\gamma}{2} - 1 \right) \|u_n\|_E^2 - c_2 \|u_n\|_2^2 + \lambda \left( 1 - \frac{\gamma}{p} \right) \|u_n\|_E^p - \frac{\lambda}{p} \|f\|_p.
\]

Divided by \( \|u_n\|_2^2 \) in (3.23), then in view of \( p \in (1, 2) \) and \( v = 0 \), we obtain

\[
(3.23) \quad \frac{1}{\|u_n\|_E^2} \left( \frac{\gamma}{2} - 1 \right) \|u_n\|_E^2 - c_2 \|v_n\|^2 + \lambda \left( 1 - \frac{\gamma}{p} \right) \|u_n\|_E^p - \frac{\lambda}{p} \|f\|_p
\]

\[
\to \frac{\gamma}{2} - 1 \text{ as } n \to \infty,
\]

showing that

\[
\frac{\gamma}{2} - 1 \leq 0,
\]

which is a contradiction with \( \gamma > 4 \). Thus \( \{u_n\} \) is bounded in \( E \). The proof is complete. \( \square \)

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